NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 3

Spring 2023

Problem 3

In how many ways can we rearrange the five letters from their envelops so that none of them reaches the correct destination?

Solution. Let $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_n$ denote the envelops of the letters $\ell_1, \ell_2, \ldots, \ell_n$, respectively. Let δ_n denote the number of ways one can rearrange these letters so that none of the letters are in their correct envelops. Let's use the term *derangement* for those arrangements where none if the letters are in the right envelop.

It is easy to see $\delta_1 = 0$ and $\delta_2 = 1$. The problem above is to identify the δ_5 (the number of possible derangements when $n = 5$).

Suppose we deranged the letters. If the envelop E_1 contains the ℓ_i for some $\mathsf{i} \in \{2, \ldots, n\}$ (there are $n-1$ different choices for i), then let's consider what is inside the envelop E_i. There are two disjoint cases:

If E_i contains a letter other than ℓ_1 : In this case, the letters other than ℓ_i is distributed among envelops, so that E_j does not contain ℓ_j for $j \neq \mathsf{i}$, and E_i does not contain ℓ_1. This is equivalent to the derangement problem with $n-1$ letters. Hence, there are δ_{n-1} possibilities.

If E_i contains the letter ℓ_1 : In this case the letters other than ℓ_1 and ℓ_i is distributed among envelops other than E_1 and E_i so that none of them are in their correct envelop. This is equivalent to a derangement problem with $n-2$ letters. Hence, there are δ_{n-2} possibilities.

From these observations, we get the inductive formula

$$\delta_n = (n-1)(\delta_{n-1} + \delta_{n-2})$$

for $n \geq 2$. Thus, we get $\delta_3 = 2$, $\delta_4 = 9$, and therefore, $\delta_5 = 4(2+9) = 44$.