NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 3

Fall 2022

Problem. The function Γ assigns a pair of natural numbers a rational number using the rules:
(i) $\Gamma(0,0) = 1$
(ii) $\Gamma(m+1,n) = 128 \cdot \Gamma(m,n)$
(iii) $\Gamma(m,n+1) = \Gamma(m,n)/32$.
Find all pairs (m,n) of natural numbers such that $\Gamma(m,n) = 2$.

Solution. Using the given rules, we deduce that $\Gamma(m,n) = 128^m 32^{-n}\Gamma(0,0) = 2^{7m-5n}$. Thus, we need to find all pairs (m,n) such that $7m - 5n = 1$. One such pair is $(3,4)$ as

$$7 \cdot 3 - 5 \cdot 4 = 1. \tag{1}$$

Suppose, there is another pair (m,n) such that

$$7m - 5n = 1, \tag{2}$$

then by subtracting (1) from (2), we get

$$\begin{aligned} 7(m-3) - 5(n-4) &= 0 \\ \Rightarrow \qquad 7(m-3) &= 5(n-4). \end{aligned}$$

Since 5 and 7 are distinct primes, 7 divides $(n-4)$ and 5 divides $m-3$. Suppose $n-4 = 7k$ (i.e. $n = 4+7k$), for some natural number k, then we get

$$m - 3 = 5(n-4)/7 = 5k$$

for the same natural number k. Thus, every pair $(m,n) = (3+5k, 4+7k)$, where k is a natural number, is a solution to $\Gamma(m,n) = 2$.