NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 7

Fall 2022

Problem 7

For a positive integer $\mathbf{n}$, let $S_{\mathbf{n}}$ be the subset of $\{1, 2, \ldots, \mathbf{n}\}$ which consists those numbers $\mathbf{k}$ for which 1 is the only common divisor of $\mathbf{n}$ and $\mathbf{k}$. Let $\alpha(\mathbf{n})$ denote the average of the numbers in $S_{\mathbf{n}}$. For example when $\mathbf{n} = 14$ then $S_{14} = \{1, 3, 5, 9, 11, 13\}$ and

$$\alpha(14) = \frac{1 + 3 + 5 + 9 + 11 + 13}{6} = 7.$$

Find a formula for $\alpha(\mathbf{n})$ in terms of $\mathbf{n}$ and justify your answer.

Solution. The formula for α is

$$\alpha(\mathbf{n}) = \frac{\mathbf{n}}{2}$$

and the key idea behind the proof is the following fact:

If the divisor of $\mathbf{n}$ and $\mathbf{k}$ is 1 then the common divisor of $\mathbf{n}$ and $\mathbf{n} - \mathbf{k}$ is also 1.

This means if $\mathbf{k}$ is in $S_{\mathbf{n}}$ then so is $\mathbf{n} - \mathbf{k}$. Further, when $\mathbf{k}$ is in $S_{\mathbf{n}}$ then $\mathbf{k}$ cannot equal $\mathbf{n} - \mathbf{k}$, otherwise 1 is not the only common divisor. Thus, the set S_{n} (for $\mathbf{n} > 2$) consists of r many pairs

$$S_{\mathbf{n}} = \{\mathsf{k}_1, \mathbf{n} - \mathbf{k}_1\} \cup \{\mathsf{k}_1, \mathbf{n} - \mathbf{k}_1\} \cup \cdots \cup \{\mathsf{k}_r, \mathbf{n} - \mathbf{k}_r\}$$

for some positive integer r and

$$\begin{aligned}
\alpha(\mathbf{n}) &= \frac{(\mathsf{k}_1 + \mathbf{n} - \mathbf{k}_1) + \cdots + (\mathsf{k}_r + \mathbf{n} - \mathbf{k}_r)}{2r} \\
&= \frac{\overbrace{\mathbf{n} + \cdots + \mathbf{n}}^{r}}{2r} \\
&= \frac{r\mathbf{n}}{2r} \\
&= \frac{\mathbf{n}}{2}.
\end{aligned}$$