NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 6

Spring 2023

Problem 6

Consider a sequence of points $P_1, P_2, P_3, P_4, \ldots$ in a plane such that P_1, P_2, P_3 are not on a straight line, and for every $n \geq 4$, P_n is the midpoint of P_{n-3} and P_{n-2}. Show that P_9 always lies on the line joining P_1 and P_5.

Solution. Suppose $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are points in a plain then any point on the line joining A and B can be expressed as

$$(1 - \mathtt{t})A + \mathtt{t}B := (((1 - \mathtt{t})a_1 + \mathtt{t}b_1, (1 - \mathtt{t})a_2 + \mathtt{t}b_2),$$

for some real number $\mathtt{t}$. In particular, when $\mathtt{t} = 0$ we are at A, when $\mathtt{t} = 1$ we are at B, and at $\mathtt{t} = \frac{1}{2}$ we are at the midpoint of A and B. Using this notation, the given condition is

$$P_n = \frac{P_{n-2} + P_{n-3}}{2}$$

for $n \geq 4$. Thus,

$$\begin{aligned}
P_9 &= \frac{P_7 + P_6}{2} \\
&= \frac{\frac{P_5 + P_4}{2} + \frac{P_4 + P_3}{2}}{2} \\
&= \frac{\frac{P_5 + P_3}{2} + P_4}{2} \\
&= \frac{P_5 + P_3 + 2P_4}{4} \\
&= \frac{P_5 + P_3 + P_1 + P_2}{4} \\
&= \frac{P_5 + P_1 + (P_3 + P_2)}{4} \\
&= \frac{P_5 + P_1 + 2P_5}{4} = \frac{3}{4}P_5 + \frac{1}{4}P_1
\end{aligned}$$

is a point on the line joining P_1 and P_5. ■