NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 7

Fall 2021

Problem 7.

Let p be a polynomial with integer coefficients. If $p(0)$ and $p(1)$ are odd, show that p has no integer roots.

Solution.

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for integers a_n, $a_{n-1}, \ldots,$ a_1, a_0. Then $p(0) = a_0$ and $p(1) = a_n + a_{n-1} + \cdots + a_1 + a_0$ are odd numbers. Thus, $a_n + a_{n-1} + \cdots + a_1$ is even.

Let r be any integer. It is easy to see that since $a_n + a_{n-1} + \cdots + a_1$ is even, then $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r$ is also even. Thus, $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$ odd. Hence, r is not a root of p. The result follows.