NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Fall 2021

Problem 8.

Let a_1, a_2, a_3, $\cdots$ be a sequence on nonnegative real numbers such that for every $n \geqslant 1$ we have

$$a_{n+2} \leqslant \frac{a_{n+1} + a_n}{n+2}.$$

Show that the series $\sum_{n=1}^{\infty} a_n$ converges.

Solution.

We show by induction that for $n \geqslant 3$ we have $a_{n+2} \leqslant \frac{a_2+a_1}{2^{n-1}}$ which will finish the proof. Indeed, for $n = 2$, we have

$$a_4 \leqslant \frac{a_3 + a_2}{4} \leqslant \frac{\frac{a_2+a_1}{3} + a_2}{4} \leqslant \frac{\frac{4}{3}(a_2 + a_1)}{4} = \frac{a_2 + a_1}{3},$$

and so for $n = 3$

$$a_5 \leqslant \frac{a_4 + a_3}{5} \leqslant \frac{2(a_2 + a_1)}{15} \leqslant \frac{a_2 + a_1}{4}.$$

For $n \geqslant 4$ we have

$$a_{n+2} \leqslant \frac{a_{n+1} + a_n}{n+2} \leqslant \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}}\right) \frac{a_2 + a_1}{n+2} = \frac{3(a_2 + a_1)}{2^{n-2}(n+2)} \leqslant \frac{a_2 + a_1}{2^{n-1}},$$

finishing the proof.