NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 2

Spring 2021

Problem 2.

Let n be a natural number such that $n+1$ is divisible by 24. Prove that the sum of all divisors of n is also divisible by 24.

Solution.

$n+1$ is divisible by 24 if and only if it is divisible by 3 and 8. Which is equivalent to

$$n \equiv -1 \pmod 3 \qquad \text{and} \qquad n \equiv -1 \pmod 8.$$

The divisors of n can be grouped in pairs a and $\frac{n}{a}$, and since no perfect square is congruent to -1 modulo 3, we must have $a \neq \frac{n}{a}$.

From

$$a\left(\frac{n}{a}\right) \equiv -1 \pmod 3$$

we conclude a and $\frac{n}{a}$ are equal to 1 and 2, in some order, modulo 3. Thus, $a + \frac{n}{a}$ is divisible by 3 and then so is the sum of all divisors of n.

From

$$a\left(\frac{n}{a}\right) \equiv -1 \pmod 8$$

we conclude a and $\frac{n}{a}$ are equal to 1 and 7, or 3 and 5, in some order, modulo 8. Thus, $a + \frac{n}{a}$ is divisible by 8 and then so is the sum of all divisors of n.

Since the sum of all divisors of n is divisible by 3 and 8, it is divisible by 24, concluding the proof.