Finite Fields – A Possible Way to Avoid Infinities in Physical Computations

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Infinities in physics: a problem. While modern physics has many successes, there are still many cases when known equations leads to meaningless infinite values for physical quantities. For example, if we compute the overall energy of an electron – including the energy $E = m_0 \cdot c^2$ corresponding to its mass m_0 and the energy the electron's electromagnetic field – we get infinity.

There are tricks – called *renormaliztaion* – that enables physicists to avoid infinities: e.g., we can assume that m_0 is close to $-\infty$ and tend to a limit. However, it is desirable to avoid infinities without adding special tricks.

Finite fields: a possible approach. Many physical quantities are discrete. For example, electric charge can only be proportional to the electron's charge – i.e., is described by an integer. For charges, addition makes physical systems: when we bring two objects together, their charges add. However, it does not necessarily mean that we need to consider infinities: for example, for an electric meter, once the number reaches a certain threshold, it turns back to 0. In general, for any prime number p, remainders modulo p with the usual addition-modulo-p and multiplication-modulo-p operations form what in mathematics is called a finite field, with usual relation between addition and multiplication and with the possibility of dividing by any non-zero number. The set of all such remainders is usually denoted by $Z/pZ = \{0, 1, \ldots, p-1\}$.

What we do in this talk. In this talk, we analyze how this idea affects the usual commonsense division of numbers into small (S), medium (M), and large (L). For example, we can consider all values < 0.1 as small, all values > 10 as large, and all others as medium.

In general, commonsense implies that if x is small, then $1/x$ is large, and vice versa. For a usual real line, no matter what thresholds we choose, some numbers are so large that they cannot be represented as a product of two small or medium numbers: in the above example, such is any number larger than 100. Interestingly, in the finite field case, this conclusion is no longer valid.

Proposition. *Suppose that Z/pZ is divided into three disjoint sets S, M, and L, for which, for every x, $x \in S$ if and only if $1/x \in L$. Then, every element $x \in L$ can be represented as a product $x = a \cdot b$ of two numbers $a, b \in S \cup M$.*

Proof. Number 1 cannot be small, since then we would have $1/1 = 1 \in L$ but $S \cap L = \emptyset$. Similarly, 1 cannot be large, so $1 \in M$. So, $\le p-1$ elements are small or large. Small and large numbers are in 1-1 correspondence via $x \mapsto 1/x$, and a number cannot be both small and large, so the number of large numbers is $\le (p-1)/2$. Thus, the number of small or medium numbers is at least $p - (p-1)/2 = (p+1)/2$.

Let us take any large number ℓ and let us consider ratios ℓ/x for all $x \in S \cup M$. There are $\ge (p+1)/2$ numbers in $S \cup M$, so we will have $\ge (p+1)/2$ different ratios. These ratios cannot be all large, since there are $\le (p-1)/2$ large numbers. Thus, at least of these ratios ℓ/x_0 is in $S \cup M$. For this ratio, we have the desired representation $\ell = x_0 \cdot (\ell/x_0)$.