Seminar 1 Problems

1. Construct a model where the following formulas are not true.

   (a) $\Box\bot$

   (b) $\Box\Box\bot$

   (c) $\Diamond p \to \Box p$

   (d) $\Diamond\Box p \to \Box\Diamond p$

2. $M, s \models \Diamond p$ iff $\exists t \in R(s) = \{x \in M \mid sRx\}$ such that $M, t \models p$

3. Is there a model where $\Box\top$ is not true?

4. Let $M = (X, R, v)$ be a model where R is transitive (if xRy and yRz then xRz). For every $x \in X$ and every modal formula, φ, prove:

   (a) $M, x \models \Box\varphi \;\Rightarrow M, y \models \varphi \quad \forall y \text{ s.t. } y \in R(x)$

   (b) $M, x \models \varphi \;\Rightarrow M, y \models \Diamond\varphi \quad \forall y \text{ s.t. } x \in R(y)$

5. Construct a reflexive ($\forall x\ xRx$) model where $\Box(\Box p \to p) \to \Box p$ is not true.

6. Show the following (Show that a model where the LHS is true doesn't necessarily have that the RHS is true):

   (a) $\Box(p \to q) \not\models p \to \Box q$

   (b) $p \to \Box q \not\models \Box(p \to q)$

7. (See text example 1.15) Give an interpretation for the following formula within the context provided: $[\pi^*](\phi \to [\pi]\phi) \to (\phi \to [\pi^*]\phi)$

   Here π denotes a program, π^* denotes a program that executes pi after a finite (possibly zero) number of times, $[\pi]\phi$ is read as 'every execution of π from the present state leads to a state bearing the information ϕ'

Seminar 2 Problems

1. Show the following are valid in all frames:

   i $\Box\top$

   ii $\Box(p \to q) \longrightarrow (\Box p \to \Box q)$

   iii $\Box(p \wedge q) \longleftrightarrow (\Box p \wedge \Box q)$

   iv $\Diamond(p \vee q) \longleftrightarrow (\Diamond p \vee \Diamond q)$

2. Show the following are not valid in all frames:

   i $\Box p \to p$

   ii $\Box p \to \Box\Box p$

   iii $\Diamond p \to \Box p$

   iv $\Box(\Box p \to p) \to \Box p$

3. Prove for any frame and any modal formulas φ, ψ

   i If $F \models \varphi$ and $F \models \varphi \to \psi$ then $F \models \psi$

ii If $F \models \varphi$ then $F \models \Box\varphi$

4. Show if R is reflexive and transitive with $S \subseteq R$ then $S^* \subseteq R$.
5. For which of the following frames is the modal formula $\Diamond\Box p \wedge (\Box(\Box p \rightarrow p)) \rightarrow \Box p$ valid: $(\mathbb{R}, <), (\mathbb{Q}, <), (\mathbb{N}, <), (\omega + 1, <)$
6. Let $R, S \subseteq X^2$. Show $(R^{-1} \circ (R \circ S)^c) \cap S = \emptyset$
7. A relation is called a strict order if it is transitive, asymmetric (a relation R is called asymmetric if whenever $(x, y) \in R$ then $(y, x) \notin R$) and irreflexive. Show that a relation which is transitive and irreflexive is a strict order.

Seminar 3 Problems

Just because there are fewer problems doesn't mean we care less.

1. Determine the minimal/maximal and maximum/minimum elements of $\{p \in \mathbb{N} \ : \ p \text{ is prime}\}$ as a subset of the poset $(\mathbb{N}, |)$ (here $|$ denotes the relation "divides").
2. Let F be a preorder. Show $F \models \Box\Box p \longleftrightarrow \Box p$

Seminar 4 Problems

1. Determine whether the following frames are well-founded, Noetherian, both or neither?
   - i $(\mathbb{P}(\mathbb{N}), \subseteq)$
   - ii $(\mathbb{R}, \leq)$
   - iii $(\mathbb{Z}^-, \leq)$
   - iv Finite strings of 0's and 1's ordered lexicographically (this is the fancy word for the dictionary order)
   - v $(A, \leq)$ where $A \subseteq \mathbb{N}$
2. Find a model $M = (X, R, v)$ and a point $x \in X$ such that $M, x \models \Box(\Box p \rightarrow p) \rightarrow \Box p$ but $M, x \not\models \Diamond p \rightarrow (\Diamond(p \wedge \neg\Diamond p))$
3. $F \models \Box\Box p \rightarrow \Box p$ iff $F \models \Diamond p \rightarrow \Diamond\Diamond p$
4. Let $F = (X, R)$ be a frame that validates the Gödel-Löb formula. Show $F \models \Diamond\Diamond p \rightarrow \Diamond p$
5. Consider the frame $F = (X, \neq_X)$. Here $\neq_X = \{(a, b) \in X^2 \ : \ a \neq b\}$.
   (a) Are the following statements true:
      i. $F \models \Diamond\Diamond p \rightarrow (\Diamond p \vee p)$
      ii. $F \models p \rightarrow \Box\Diamond p$
   (b) When are the following statements true:
      i. $F \models \Diamond\Diamond p \rightarrow \Diamond p$
      ii. $F \models \Diamond p \rightarrow \Diamond\Diamond p$
      iii. $F \models \Box(\Box p \rightarrow p) \rightarrow \Box p$ (Hint: What types of frames validate this formula?)

Bonus: (This is the remaining part of the proof that was not finished) Prove: If $F \models \Diamond p \rightarrow \Diamond(p \wedge \neg\Diamond p)$ then F is a strict Noetherian poset.

Seminar 5 Problems

1. Let $F = (T, R_F, R_P)$ be any frame.

   i Show $F \models q \to [P]\langle F\rangle q$ iff $\forall s \forall t (sR_P t \Rightarrow tR_F s)$

   ii Suppose $R_F = R_P^{-1}$. Show $F \models \langle P\rangle q \to [F]\langle P\rangle q$ iff R_P is transitive.

2. Let R be a relation on the set X.

   i Show $R \subseteq R \circ R^{-1} \circ R$ if $X = \{y \in X \mid \exists x \in X \; xRy\}$.

   ii When is $\Delta_X \subseteq R \circ R^{-1}$? (Recall $\Delta_X = \{(x, x) | x \in X\}$)

   iii When is $\Delta_X \subseteq R^{-1} \circ R$?

3. Let S and T be equivalences on the same set. Show $S \circ T = T \circ S$ iff $S \circ T$ is an equivalence relation.

4. Consider the bimodal frame $F = (X, R, X^2)$. Find a modal formula φ such that:

   i $F \models \varphi$ iff F satisfies $\forall x \exists y (yRx)$

   ii $F \models \varphi$ iff F satisfies $\exists x \neg \exists y (xRy)$

5. Let D be the operator where $M, x \models D\varphi$ iff $\exists y \in X \setminus \{x\}$ s.t. $M, y \models \varphi$. Write $E\varphi$ in terms of D (where E is the universal diamond). Can we write $D\varphi$ in terms of E?

6. Let $F = (T, R_F, R_P)$ be a temporal frame. How does one interpret truth of the formula $[P]q \wedge q \wedge [F]q$ at a point?

Seminar 6 Problems

1. Let R, S be equivalence relations on X. Show $R \circ S = X^2$ iff $S \circ R = X^2$.

2. Let X be a set, $R \subseteq X^2$. Fix a valuation ,v, on X. For any $x \in X$ and for any formula φ show:

   i. $(X, R^*, v), x \models \Box\varphi \;\Rightarrow\; (X, R, v), y \models \Box\varphi \quad \forall y \in R(x)$

   ii. $(X, R, v), x \models \Diamond\varphi \;\Rightarrow\; (X, R^*, v), y \models \Diamond\varphi \quad \forall y \in R^{-1}(x)$

3. Consider a temporal frame (T, R_F, R_P) (Recall a temporal frame is one such that $R_F = R_P^{-1}$, R_F and R_P are both transitive and irreflexive). Is it true that $R_F \circ R_P = X^2$.

4. Let (X, R) be a frame. Interpret the following for that frame (here A is the universal box and E is the universal diamond):

   i. $p \to E\Diamond p$

   ii. $Ep \to p$

   iii. $Ep \to \Diamond p$

   iv. $E\Box\top$

   v. $p \to AEp$

   vi. $p \to Ep$

5. Let $R = \{(x, x+1) \mid x \in \mathbb{Z}\}$. What is R^*?

6. Let $F = (T, R_F, R_P)$ be a temporal frame such that R_P is a strict total order. Show:

$$F \models (q \wedge [P]q) \to \langle F\rangle[P]q \iff \forall x \exists y \; (xR_F y \text{ and } \neg\exists z \; (xR_F zR_F y))$$

Seminar 7 Problems

1. Find the supremum and infimum of the following subsets of $\mathbb{R}$, (if they exist):

   i. $\{\frac{1}{n} \mid n \in \mathbb{N},\ n \neq 0\}$

   ii. $(0, 1)$

   iii. $\mathbb{N}$

   iv. $\{(-1)^n - \frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$

2. Let $A, B \subseteq \mathbb{R}$ such that both A and B are bounded above. What is $\sup\{a + b \mid a \in A,\ b \in B\}$

3. Let (X, R) be a poset with the least upper bound property. Show that for any subset $S \subseteq X$, if S is bounded below then $\inf(S)$ exists.

4. A strict total order $(X, <)$ is said to be continuous if for every pair of nonempty, disjoint sets U, V such that $U \cup V = X$ and $\forall x \in U$, $\forall y \in V$ $x < y$, there exists $z \in X$ such that $\forall x \in U \forall y \in V$ $x \leq z \leq y$.

   Let $F = (X, <)$ be a strict linear order. Show that F is continuous iff every nonempty subset of X with an upperbound has a supremum.

Seminar 8 Problems

1. Let X be a set and R a partial order on X. Suppose there are $a, b \in X$ such that $(a, b) \notin R$ and $(b, a) \notin R$ (that is, R is not a linear order). Show that S^* extends R where $S = R \cup \{(a, b)\}$.

2. Show $\mathbb{R}[x]$ is a vector space over $\mathbb{R}$.

3. Let V be a vector space over $\mathbb{R}$ and $U \subseteq V$ be a linearly independent set. If $v \notin span(U)$, show that $U \cup \{v\}$ is a linearly independent set.

4. Let (V, R) be a frame with $V = \{v_1, \cdots, v_n\}$ a finite set. The adjacency matrix $A = (a_{ij})$ of (V, R) is an n by n matrix where an element a_{ij} is 1 if $v_i R v_j$ and 0 otherwise. Find the adjacency matrices of the following:

   (a) (V, R^{-1})

   (b) $(V, R \circ S)$

   (c) (V, R^n) where $n \in \mathbb{Z}^+$

   Note: When doing operations on adjacency matrices, treat $1 + 1$ as 1.

5. Let (V, R) be a frame with V finite. What property of the adjacency matrix of (V, R) corresponds to (V, R) being symmetric? What property corresponds to (V, R) being disconnected?

6. Show that all subsets of $\mathbb{N}$ can be made "small" or "big" in the following sense.

   (a) Finite sets are small.

   (b) X is big if X^c is small.

   (c) A subset of a small set is small and a superset of a big set is big.

   (d) If X and Y are small then $X \cup Y$ is small.

   (e) If X and Y are big then $X \cap Y$ is big.

   (f) If X^c is finite then X is big.

Seminar 9 Problems

1. Is $(\mathbb{Q}, \leq)$ a generated subframe of $(\mathbb{R}, \leq)$?.

2. Let $F = (X, R)$ be a frame such that R is reflexive and Euclidean. (A relation $R \subseteq X^2$ is Euclidean if it satisfies $\forall a, b, c \in X \quad aRb \wedge aRc \rightarrow \ bRc$). Show that every two point-generated subframes of F are either disjoint or identical.

3. Let $F = (X, R)$ be a frame and φ a modal formula. Show the following are equivalent:

   i. $F \models \varphi$
   ii. $G \models \varphi$ for all generated subframes $G \underset{\rightarrow}{\subseteq} F$
   iii. $G \models \varphi$ for all rooted subframes $G \underset{\rightarrow}{\subseteq} F$

4. Show there is no set Φ of modal formulas such that $(X, R) \models \Phi \iff (X, R)$ is connected.

Bonus: Let G be a generated subframe of F. For every modal formula φ show:

   i. $\forall x \in G$ we have $G, x \models \varphi \ \Leftrightarrow \ F, x \models \varphi$
   ii. $F \models \varphi \Rightarrow G \models \varphi$

Seminar 10 Problems

1. Show there is no set Φ of modal formulas such that $(X, R) \models \Phi$ iff R is asymmetric.

2. Show there is no set Φ of modal formulas such that $(X, R) \models \Phi$ iff R is anti-symmetric.

3. Show that $(\mathbb{N}, \leq) \twoheadrightarrow (X, X^2)$ where:

   i. $X = \{0, ..., n : n \in \mathbb{N}\}$
   ii. $X = \mathbb{N}$

Turtle

He's not upset at you. He's upset I wasted my time drawing a turtle.