NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 9

Spring 2024

Problem 9

A set $S \subset \mathbb{Z}^2$ is called balanced if for each point $(x, y) \in S$, exactly two of its 4 adjacent points $(x \pm 1, y)$ and $(x, y \pm 1)$ belong to S. For example, $S = \{(0,0), (0,1), (1,1), (1,0)\}$ is balanced while $S = \{(0,0), (0,1), (0,2)\}$ is not. Find all positive integers n such that there exists a balanced set of n elements.

Solution: We first prove that if a set S is balanced, then it must contain an even number of elements. To see this, we first call a point $(x, y) \in \mathbb{Z}^2$ odd when $x+y$ is odd, and we call it even when $x+y$ is even. Notice that for any odd point, (x, y), four of its neighbours are all odd and vice versa.

Let S be a balanced set with n_1 odd points and n_2 even points. Each odd point has exactly 2 even points as its neighbour, and each even point has exactly 2 odd points as its neighbour. Therefore, if we multiply the total number of odd points n_1 by 2, we get the total number of their even points neighbours but each is counted twice. This proves that $2n_1 = 2n_2$ and thus $n_1 = n_2$ and $n = n_1 + n_2 = 2n_1$ is even.

Let us now find all even positive integers n such that there exists a balanced set of n points. One should be able to rule out $n = 2$ easily since if a balanced set is non-empty, there is at least one point with two of its neighbours, which already has at least 3 points. $n = 4$ is possible with the example given in the question.

To see $n = 6$ is impossible: suppose S is balanced with 6 elements. Without loss of generality, let the lowest point on the left-most column of S be $(0,0)$ (if not, we can always shift the set S around). From the construction, all points $(x, y) \in S$ has $x \geq 0$ since $(0,0)$ is already on the left-most column; and $(0,-1)$ is also not in S because $(0,0)$ is the lowest along that column. Therefore, both $(1,0)$ and $(0,1)$ must be in S. The point $(1,1)$ cannot be in S, otherwise, the other two points must form a balanced set on their own, which is impossible. Therefore, $(0,2)$ must be in S since among the four neighbours of $(0,1)$, $(-1,1)$ and $(1,1)$ are both not in S. Now both $(0,2)$ and $(1,0)$ are missing one neighbour. No matter which neighbour we pick for them, one can easily see that the result set is not balanced. This proves that $n \neq 6$.

Finally, we claim that any even number $n \geq 8$ is possible. For $k \geq 2$, consider the set

$$S = \{(1,0), (1,k)\} \cup \{(0,n), (2,n) : 0 \leq n \leq k\}.$$

The set S forms the boundary of a $2 \times k$ rectangle. This set contains exactly $2(k+2)$ elements, and one can verify that it is balanced. Let $k = 2, 3, 4, \ldots$, we have $n = 2(k+2) = 8, 10, 12, \ldots$. This proves that any even number $n \geq 8$ is possible.

In conclusion, $\{4\} \cup \{8, 10, 12, \ldots\}$ is the set of all n such that there exists a balanced set of n elements.