NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 6

Spring 2023

Problem 6

Let $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ be a map between positive integers such that it is strictly increasing (i.e. $f(x+1) > f(x)$) and

$$f(f(x)) = 3x$$

for all positive integers x. Find n for which $f(n) = 56$. Justify your answer.

Solution. Since $f(f(1)) = 3$, we conclude $f(1)$ cannot equal 1 because, otherwise, it leads to a contradiction $1 = f(1) = f(f(1)) = 3$. Thus, $f(1) > 1$, as any positive number other than 1 is greater than 1. Because f is monotonically increasing we have $f(1) < f(f(1)) = 3$. Thus $f(1) = 2$, and the given formula $f(f(x)) = 3x$ implies

$$f(n) = \begin{cases} 2 \cdot 3^k & \text{if } n = 3^k \\ 3^{k+1} & \text{if } x = 2 \cdot 3^k \end{cases}$$

as shown in the table

n	1	2	3	4	5	6	7	8	9	...	18	...
$f(n)$	2	3	6	?	?	9	?	?	18	?	27	...

Because of the strictly increasing condition, we notice that

$$6 < f(4) < f(5) < 9$$

which forces $f(4) = 7$ and $f(5) = 8$ as well as

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...	18	...
$f(n)$	2	3	6	7	8	9	12	15	18	?	?	21	?	?	24		27	...

The monotonicity of f means $18 < f(10) < f(11) < 21$ and $21 < f(13) < f(14) < 24$ which forces $f(10) = 19$, $f(11) = 20$, $f(13) = 22$ and $f(14) = 23$. Continuing this way, we check

$$f(29) = 56,$$

thus $n = 29$.

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