NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Spring 2023

Problem 8

Fermat's little theorem. If a is an integer coprime to n then

$$a^{\varphi(n)} \equiv 1 \mod n,$$

where $\varphi(n)$ is the Euler's totient function.

Use the result above to identify $3^{2023^{2023}}$ **mod 7**.

Solution. The Euler totient function $\varphi(n)$ equals the number of positive integers k less than n for which $\gcd(n, k) = 1$. For example,

$$\varphi(2) = 1, \quad \varphi(6) = \#\{1, 5\} = 2, \quad \varphi(7) = \#\{1, 2, 3, 4, 5, 6\} = 6,$$

so on and so forth.

When we set $a = 3$ and $n = 7$, Fermat's little theorem implies $3^6 \equiv 1 \mod 7$. Thus, we need to identify

$$2023^{2023} \mod 6$$

in order to get the answer. Here we are relying on the general formula

$$a^k \mod n = a^{k \bmod \varphi(n)} \mod n$$

which is a direct consequence of Fermat's little theorem. Using this formula iteratively, we get

$$\begin{aligned} 3^{2023^{2023}} \mod 7 &= 3^{2023^{2023} \bmod \varphi(7)} \mod 7 \\ &= 3^{2023^{2023} \bmod 6} \mod 7 \\ &= 3^{2023^{(2023 \bmod \varphi(6))} \bmod 6} \mod 7 \\ &= 3^{2023^{(2023 \bmod 2)} \bmod 6} \mod 7 \\ &= 3^{2023^{1} \bmod 6} \mod 7 \\ &= 3^{1} \mod 7 \\ &= 3. \end{aligned}$$

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