MATH 527 Masters Written/PhD Qualifying Exam Syllabus

Text: An Introduction to Analysis, Fourth Edition, by William R. Wade

- Basic properties of real numbers. Absolute value, properties starting at Definition 1.4 through the end of §1.2; Completeness Axiom, §1.3.
- Real sequences. Limits: definition, properties, basic theorems, Bolzano-Weierstrass Theorem, Cauchy sequences, limits superior/inferior, §§2.1–2.5.
- Functions on $\mathbb{R}$. Two-sided limits: $\varepsilon - \delta$ definition, sequential characterization, limit rules, Squeeze Theorem, Comparision Theorem; one-sided limits: definitions, basic theorems, extension to limits at infinity, §§3.1–3.2.

  Continuity. Definition, basic properties, compositions, Extreme Value Theorem, Intermediate Value Theorem; uniform continuity, §§3.3–3.4.
- Differentiability. Definition, basic properties and rules, left and right derivatives, Chain Rule, Mean Value Theorem (extensions and applications), §§4.1–4.3.

  Taylor's Theorem, l'Hospital's Rule and Inverse Function Theorems, §§4.4–4.5.
- Riemann Integration. Definition (upper/lower sums, upper/lower integrals), Riemann sums and connection with the definition, basic theorems/properties of the integral, Fundamental Theorem of Calculus, integration by parts, change of variables, §§5.1–5.3.

  Improper Riemann integration. Definition, local integrability, basic properties/theorems, absolute integrability and conditional integrability, §5.4.

Examples of typical homework problems—this is *not* an exhaustive list:

1. Find the sup and inf of $\left\{2 + \dfrac{1}{3n^2} : n \in \mathbb{N}\right\}$. Justify your answer.
2. Give an $\varepsilon - N$ proof that $\displaystyle\lim_{n\to\infty} \frac{2n^3 + n^2 - n}{5n^3 - 1} = \frac{2}{5}$.
3. Suppose $x_n \to 1$ as $n \to \infty$. Give an $\varepsilon - N$ proof that $\dfrac{x_n - 2}{x_n} \to -1$ as $n \to \infty$.
4. State the relations between sup/inf of a set E and $-E$.
5. Give an $\varepsilon - \delta$ proof that
$$\lim_{x\to 2} \frac{x^2+1}{x-3} = -5.$$
6. Find the $\limsup$ and $\liminf$ of the sequence
$$x_n = 3 + \frac{n^2(1 + (-1)^n)}{n+1}.$$
   Justify your answer.

7. Use the definition to prove that

$$\lim_{x\to 3^+} \frac{x+2}{x^2-2x-3} = \infty.$$

8. Use the definition to prove that

$$\lim_{x\to 4^+} \frac{3}{x^2-3x-4} = \infty.$$

9. Suppose $f : (0,\infty) \to \mathbb{R}$ is differentiable and $f'(x) \to 0$ as $x \to \infty$. Prove that if $g(x) = f(2x) - f(x)$, then $g(x) \to 0$ as $x \to \infty$.

10. Suppose $a_k > 0$ for all k,

$$\sum_{k=1}^{\infty} a_k \text{ converges}, \qquad b_n \uparrow \infty \text{ as } n \uparrow \infty \qquad \text{and } \sum_{k=1}^{\infty} a_k b_k \text{ converges}.$$

Prove that

$$\lim_{n\to\infty} b_n \sum_{k=n}^{\infty} a_k = 0.$$

11. Suppose $f, g : E \to \mathbb{R}$ are uniformly continuous. Assume that f is bounded and $\inf g(E) > 0$. Prove that f/g is uniformly continuous on E.

12. Suppose $f : [-1,1] \to \mathbb{R}$ is continuous and $\displaystyle\int_{-1}^{1} f(x)x^2\,dx = 0$. Prove that $f(x) = 0$ for at least one $x \in [-1,1]$.

13. Suppose $g_n \geq 0$ is Riemann integrable on $[0,1]$, $n \in \mathbb{N}$ and f is Riemann integrable on $[0,1]$. Prove that if $\displaystyle\lim_{n\to\infty} \int_0^1 g_n(x)\,dx \to 0$, then $\displaystyle\lim_{n\to\infty} \int_0^1 f(x)g_n(x)\,dx \to 0$.

14. Suppose $f \geq 0$ is locally integrable on $[0,1)$, $g \geq 0$ is improperly integrable on $[0,1)$, and

$$\lim_{x\to 1^-} \frac{f(x)}{g(x)} = L \in (0,\infty).$$

Prove that f is improperly integrable on $[0,1)$.

15. Suppose

$$f(x) = \begin{cases} (x-1)^2 \cos\dfrac{1}{x-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Decide whether or not f is continuous at $x = 1$ and prove your answer.

16. Prove there exists $x \in \mathbb{R}$ such that $x^3 - 1 = \sin x$.

17. Use the definition ($\varepsilon - \delta$) to prove $f(x) = 4x - 3x^2$ is uniformly continuous on the interval $(0,3)$.

18. Suppose $x_1 = 3$ and for $n \geq 1$, $x_{n+1} = 8 - \dfrac{12}{x_n}$. Prove x_n is convergent and identify the limit.

19. Prove $\tan x \geq x - \dfrac{x^3}{3}$ for $0 \leq x < \dfrac{\pi}{2}$.

20. Suppose

$$f(x) = \begin{cases} x^3 \cos \dfrac{1}{x^2} & \text{if } x \neq 0 \\ \\ 0 & \text{if } x = 0 \end{cases}$$

a) Prove $f'(0)$ exists.

b) Is $f'(x)$ continuous at $x = 0$? Justify your answer.

21. Let $f : [1, 2] \to \mathbb{R}$ be continuously differentiable and one-to-one on $[1, 2]$. Prove that

$$\int_1^2 f(x)\, dx + \int_{f(1)}^{f(2)} f^{-1}(x)\, dx = 2f(2) - f(1).$$

22. Decide if the function $f(x) = \dfrac{2 + \sin x}{x[1 + (\ln x)^{1/2}]}$ is improperly integrable on $[e, \infty)$.

23. Write as a telescoping series to find the sum of $\displaystyle\sum_{k=2}^{\infty} \log\left(\frac{(k-1)(k+3)}{k(k+2)}\right)$.

24. Give an $\varepsilon - N$ proof that $\displaystyle\lim_{n\to\infty} \frac{2n^3 + 3n^2 + 5}{6n^3 + 2n^2 + 1} = \frac{1}{3}$.

25. Suppose

$$f(x) = \begin{cases} 3x & \text{if } x \in \mathbb{Q} \\ -3x & \text{if } x \notin \mathbb{Q} \end{cases}$$

a) Prove f is continuous at $x = 0$.

b) Prove f is not continuous at $x \neq 0$.

26. Prove there exists some $x \in \mathbb{R}$ such that $x^3 + 2 = e^{-x}$.

27. Decide if the function $f(x) = \dfrac{1}{x + x^2}$ is uniformly continuous on $(0, 2)$. Justify your answer.

28. a) Suppose $f(x) = \dfrac{2}{x} - \ln x$ for $x \in (0, 1]$. Prove f^{-1} exists.

b) Find the image $f((0, 1])$ of f.

c) Prove f^{-1} is differentiable on its domain.

29. Use the Mean Value Theorem to prove $1 - x + \frac{x^2}{2} \geq \ln x$ for $0 < x \leq 1$.

30. Let $P_n = \{\frac{j}{n} : 1 \leq j \leq n\}$ and suppose $f(x) = \frac{1}{2}$ for $x \in (0,1)$ and $f(0) = 1$, $f(1) = 3$.

a) Compute the upper and lower sums of f over P_n.

b) Compute the limits as $n \to \infty$ of the upper and lower sums you computed in part a).

31. Let $f : [1,4] \to \mathbb{R}$ is continuous and for every $c \in (1,2)$,

$$\int_1^{c^2} x^2 f(x)\, dx = c^8.$$

Prove that $f(x) = 4x$ for $x \in [1,4]$.

32. Suppose $\displaystyle\int_e^{e^2} |f(x)|\, dx < e$. Prove $\left|\displaystyle\int_1^2 e^{-2x} f(e^x)\, dx\right| < e^{-2}$.

33. Prove $\inf(0,1) = 0$.

34. Find the inf and sup of the interval $[-1,2)$; justify your answer.

35. Find the inf and sup of the set $\left\{\frac{1}{n} + n : n \in \mathbb{N}\right\}$; justify your answer.

36. Find the inf and sup of $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$ and justify your answer.

37. Use the definition of the limit to prove the following.

(a) $\displaystyle\lim_{n\to\infty} \frac{3n^3 + 2n^2 + 8}{4n^3 + 9} = \frac{3}{4}$.

(b) $\displaystyle\lim_{n\to\infty} \frac{1}{5n^2 - 1} = 0$.

(c) $\displaystyle\lim_{n\to\infty} \frac{1}{6n^2 - 8n + 1} = 0$.

38. We say a sequence $a_n \to \infty$ if for each $M > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Longrightarrow a_n > M.$$

(a) State the corresponding definition for $a_n \to -\infty$.

(b) Prove that if $x_n \to \infty$, then (x_n) has a minimum; that is, for some $N \in \mathbb{N}$, $x_n \geq x_N$ for all $n \in \mathbb{N}$.

39. Suppose that the sequence (a_n) is increasing and it has a convergent subsequence. Does this imply that (a_n) is convergent? Prove or provide a counterexample.

40. Is there any sequence (a_n) which has no convergent subsequence, but $(|a_n|)$ is convergent? Justify your answer.

41. Prove that if (a_n) is bounded, and all of its convergent subsequences converge to a, then (a_n) also converges to a.

42. Prove or provide a counterexample:

   (a) If (x_n) Cauchy and (y_n) is bounded, then $(x_n y_n)$ is Cauchy.

   (b) If (x_n) and (y_n) are Cauchy and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\dfrac{x_n}{y_n}$ is Cauchy.

43. Give $\varepsilon - \delta$ proofs of the following limits.

   (a) $\lim_{x\to 2}(3x^2 - 5x + 1) = 3$

   (b) $\lim_{x\to 3} \dfrac{x-4}{x-2} = -1$

   (c) $\lim_{x\to 2} \dfrac{x^2+1}{x-3} = -5$

   (d) $\lim_{x\to 1} \dfrac{2x-5}{x-3} = \dfrac{3}{2}$

44. Prove the following limits do not exist.

   (a) $\lim_{x\to -5} \dfrac{|x+5|}{x+5}$

   (b) $\lim_{x\to 1} \sin \dfrac{1}{x-1}$

45. State the definition of the following.

   (a) $\lim_{x\to -\infty} f(x) = -\infty$

   (b) $\lim_{x\to a} f(x) = -\infty$

46. Determine the limits (using methods you learned in Calculus I) and then use the definition to prove your answer.

   (a) $\lim_{x\to\infty} \dfrac{x^3+2}{1-2x^3}$

   (b) $\lim_{x\to -\infty} \dfrac{x-3}{x+4}$

   (c) $\lim_{x\to -\infty} \dfrac{x^2-1}{x^2-3}$

   (d) $\lim_{x\to 4^+} \dfrac{1}{x^2-9x+20}$

   (e) $\lim_{x\to\infty} (x - 3x^2)$

   (f) $\lim_{x\to\infty} \dfrac{1}{x^2-x-6}$

47. Prove that the equation $e^x = 2\cos x + 1$ has at least one solution in $\mathbb{R}$.

48. Prove that if f is continuous on $[a, b]$ with $f(x) > 0$ for all $x \in [a, b]$, then $1/f$ is bounded on $[a, b]$.

49. Give a short justification for your answer to each of the following questions. When applicable, a sketch will suffice.

   (a) Is it possible for a continuous function f defined on $[0,1]$ to satisfy $f([0,1]) = (0,1)$?

   (b) Is it possible for a continuous function f defined on $(0,1)$ to satisfy $f((0,1)) = [0,1]$?

50. Let $f : [0,1] \to [0,1]$ be continuous. Prove that f has at least one fixed point in $[0,1]$; that is, there exists $x \in [0,1]$ such that $f(x) = x$.

    HINT: Consider the function $g(x) = f(x) - x$ and apply the Intermediate Value Theorem.

51. Prove that the function $f(x) = 1/x$ is not uniformly continuous on $(0,1)$.

52. Suppose $f : A \to \mathbb{R}$ is uniformly continuous on A. Prove that if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence.

53. Show $f(x) = 1/x^2$ is uniformly continuous on $[1,\infty)$, but not on $(0,1)$.

54. Let $f(x) = x|x|$. Prove that

$$\lim_{h\to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} = 0$$

    but $f''(0)$ does not exist.

55. Suppose $f, g : A \to \mathbb{R}$, where A is *any* nonempty set. For example, A could be a set of real numbers, or A could even be sets of sets (for instance, the set of all partitions of an interval $[a,b]$). Prove that

$$\sup_A (f+g) \le \sup_A f + \sup_A g \qquad \inf_A (f+g) \ge \inf_A f + \inf_A g \qquad .$$