**Basic Real Analysis Comprehensive Exam Syllabus**

## Math 528

Basic Reference: An Introduction to Analysis, Fourth Edition, by William R. Wade

- Sequences of functions. Pointwise and uniform convergence, uniform Cauchy criterion, continuity, differentiation and integration of limits of sequences of functions. Series of functions: uniform convergence, continuity of the sum, termwise integration and differentiation, Weierstrass $M$-test, §§7.1–7.2.

  Power series. Radius of convergence, interval of convergence, properties, Abel's Theorem. Analytic functions: definition, uniqueness; $C^\infty$ functions and sufficient conditions for analyticity, §§7.3–7.4.

- Euclidean spaces. Euclidean norm $\|\cdot\|$ and basic properties: triangle inequality, dot product and Cauchy-Schwarz Inequality; other equivalent norms: $\ell^1$ norm $\|\cdot\|_1$, sup norm $\|\cdot\|_\infty$, §8.1.

  Convergence in $\mathbb{R}^n$. Usual limit theorems from $\mathbb{R}$ continue to hold; convergence in $\mathbb{R}^n$ is equivalent to componentwise convergence in $\mathbb{R}$. Sequences in $\mathbb{R}^n$ converge iff they are Cauchy; the Bolzano-Weierstrass Theorem holds in $\mathbb{R}^n$. §9.1.

  Limits of functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}$. Usual limit theorems hold. Prove limits do not exist by taking different paths of approach that give different limits. §9.3 Examples 9.18–9.21.

- Metric spaces. Basic definition, examples, subspaces, open balls, open sets, closed sets. Convergence of sequences in a metric space, boundedness, completeness, sequential characterization of a set being closed, cluster points, limits of functions, continuity, basic limit/continuity rules, §§10.1–10.2.

  Basic topology. Open sets, closed sets and their behavior with regards to unions, intersections and complements; interior, closure, boundary: keep in mind the useful tool that the interior of a set is the largest open set contained in that set and the closure is the smallest closed set containing that set; closure, interior and boundary of unions and intersections; relations between boundary/interior/closure, §10.3.

  Compact sets. Definition, properties in terms of being closed and bounded; separable metric spaces, Lidelöf's Theorem, the Heine-Borel Theorem; continuous functions on compact spaces are uniformly continuous, §10.4.

  Connected sets. Definition, relatively open/closed, pathwise connected; characterization of connected sets in $\mathbb{R}$, §10.5.

  Continuity. Alternate definition in terms of inverse images; continuous functions and images of compact sets and connected sets, Extreme Value Theorem, §10.6.

  Stone-Weierstrass Theorem. Algebras of continuous functions, uniformly closed, uniformly dense, separates points, §10.7.

- Differentiation in $\mathbb{R}^n$. Partial derivatives, function spaces $C^p(V)$; changing order of mixed partials, definition of differentiability for $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$; form of the derivative when it exists; usual derivative rules hold, including differentiability at a point implies continuity there; proving differentiability and non-differentiability via the definition; sufficient conditions that imply differentiability, §§11.1–11.2.

Higher dimensional versions of the Mean Value Theorem and applications, §11.5, Theorem 11.30 through Corollary 11.35.

Examples of typical homework problems—this is *not* an exhaustive list:

1. Suppose $g$ is continuous on $[a,b]$, $\inf_{[a,b]} g > 0$ and $g_n \to g$ uniformly on $[a,b]$. Prove that for some $N \in \mathbb{N}$, $\inf_{[a,b]} g_n > 0$ for all $n \geq N$.

2. Suppose $f$ and $g$ are bounded on $E$ and $f_n \to f$, $g_n \to g$, both uniformly on $E$. Then $f_n g_n \to fg$ uniformly on $E$.

3. Recall a sequence $f_n : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$, is uniformly bounded if there is $M > 0$ such $|f_n(x)| \leq M$ for all $x \in E$ and $n \in \mathbb{N}$.

   Suppose for each $n \in \mathbb{N}$, $f_n$ is bounded. If $f_n \to f$ uniformly on $E$ then $\{f_n\}$ is uniformly bounded and $f$ is a bounded on $E$.

4. Let $E$ be a nonvoid subset of $\mathbb{R}$ and suppose $f_n : E \to \mathbb{R}$ are uniformly continuous and converge uniformly to $f$ on $E$. Prove $f$ is uniformly continuous on $E$.

5. Prove
$$\int_0^{\pi/2} \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

6. Prove the following are analytic on the indicated set and find the Maclaurin expansion.

   (a) $x^2 + \cos x$, $(-\infty, \infty)$

   (b) $\dfrac{x^2}{1+x^5}$, $(-1,1)$.

7. Decide if the following are metrics on $\mathbb{R}$:

   (a) $d(x,y) = (x-y)^2$

   (b) $d(x,y) = \sqrt{|x-y|}$

   (c) $d(x,y) = |x-2y|$

   (d) $d(x,y) = |x^2 - y^2|$.

8. Let $C[a,b]$ be the set of continuous functions on the interval $[a,b]$. For $f, g \in C[a,b]$ define
$$d(f,g) = \int_a^b |f(x) - g(x)|\, dx.$$
   Prove this is a metric on $C[a,b]$.

9. Let $E^\circ$ be the interior of $E$.

   (a) Prove $E$ is open iff $E = E^\circ$.

(b) Prove $(E^\circ)^c = \overline{E^c}$

(c) Do $E$ and $\overline{E}$ always have the same interiors?

(d) Do $E$ and $E^\circ$ always have the same closures?

10. Let $\partial E$ denote the boundary of $E$.

    (a) Prove $E$ is closed iff $\partial E \subseteq E$.

    (b) Prove $E \cup \partial E = \overline{E}$.

    (c) Show there are sets $A, B \subseteq \mathbb{R}$ such that $\partial(A \cup B) \neq (\partial A) \cup (\partial B)$.

    (d) Show there are sets $A, B \subseteq \mathbb{R}$ such that $\partial(A \cap B) \neq (\partial A) \cap (\partial B)$.

    (e) Prove $\partial E = \overline{E} \backslash E^o$.

    (f) Prove $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $(A \cap B)^\circ = A^\circ \cap B^\circ$.

11. Suppose $X$ is a metric space and $A, B \subseteq X$ are compact. Prove $A \cap B$ and $A \cup B$ are compact.

12. Suppose that $A \subseteq \mathbb{R}$ is nonvoid and compact. Prove $\inf A \in A$.

13. Prove that if $A \subseteq \mathbb{R}$ is connected, then so is $A^\circ$.

14. Prove that $\partial(A \cap B) \cap (A^c \cup (\partial B)^c) \subseteq \partial A$.

15. Prove that if $\{p_n\}$ is Cauchy in the metric space $X$ and some subsequence converges, then $\{p_n\}$ converges.

16. Equip $\mathbb{R}$ with the discrete metric. Prove the resulting metric space is complete.

17. Prove $f^{-1}(E^c) = (f^{-1}(E))^c$.

18. Let $X$, $Y$ be metric spaces and suppose $f : X \to Y$ is continuous. Prove $f(\overline{E}) \subseteq \overline{f(E)}$.

19. Let $X$ be a metric space and let $f : X \to \mathbb{R}$ be continuous. Let $Z = \{x \in X : f(x) = 0\}$. Prove $Z$ is closed.

20. Suppose $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$. Prove the following identity for the inverse images:
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

21. Use the previous problem to give an EASY proof of the composition rule: Let $X$, $Y$,and $Z$ be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions then $g \circ f : X \to Z$ is also continuous.

22. Evaluate the limit
$$\lim_{n\to\infty} \int_0^3 \sqrt{x + 1 + \sin\frac{x}{n}}\, dx$$
and justify your answer.

23. Let $X$ be a metric space and suppose $f_n : X \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to some $f : X \to \mathbb{R}$. Prove that if $\{x_n\}$ is a sequence of points in $X$ that converges to some $x \in X$, then $f_n(x_n) \to f(x)$.

24. Suppose $K$ is a compact metric space and $E$ is a countable dense subset of $K$. Given $\delta > 0$, prove there are $x_1, \dots, x_n \in E$ such that

$$K \subseteq N_\delta(x_1) \cup \cdots \cup N_\delta(x_n).$$

25. Suppose $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R \in (0, \infty)$.

  (a) Find the radius of convergence of $\sum_{k=0}^{\infty} a_k x^{2k}$.

  (b) Find the radius of convergence of $\sum_{k=0}^{\infty} a_k^2 x^k$.

26. Suppose $\{a_k\}$ is a bounded sequence of real numbers. Prove $\sum_{k=0}^{\infty} a_k x^k$ has a positive radius of convergence.

27. Prove that

$$f(x) = \sum_{k=0}^{\infty} \left( \frac{x}{(-1)^k + 4} \right)^k$$

is differentiable on $(-3, 3)$ and give a series expansion for the derivative.

28. Decide if the limit exists. If it does, find it. Justify your answer.

  (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{(x^2+y^2)^{1/3}}$.

  (b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^2+2y^4}$.

  (c) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin x \sin y}{x^2+y^2}$.

29. Compute $f_x$ and find where it is continuous.

  (a) $f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

  (b) $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{(x^2+y^2)^{1/3}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

30. Prove

$$f(x,y) = \begin{cases} \dfrac{x^3-xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

is continuous on $\mathbb{R}^2$, has first-order partials on $\mathbb{R}^2$, but fails to be differentiable at the origin.

31. Prove that the first partials of $f(x,y) = (xy)^{1/5}$ exist at $(x,y) = (0,0)$, but $f$ is not differentiable there.

32. Let $u(x,t) = \dfrac{e^{-x^2/4t}}{\sqrt{4\pi t}}$, $\quad t > 0, \quad x \in \mathbb{R}$. If $a > 0$, show $u(x,t) \to 0$ as $t \to 0^+$, uniformly for $x \in [a, \infty)$.

33. Show that
$$f(x) = \sum_{n=1}^{\infty} \left[1 + x^{2n} - \sqrt{1 + x^{4n}}\right]$$
is differentiable on $(-1,1)$ and give a series expansion of the derivative.

34. Suppose $H = [a,b] \times [c,d]$, $f : H \to \mathbb{R}$ is continuous and $g : [a,b] \to \mathbb{R}$ is integrable. Prove that $F(y) = \int_a^b g(x) f(x,y)\, dx$ is uniformly continuous on $[c,d]$.

35. Let $E = \{(x,y) : x > 0, y > 0\}$ and $\mathbf{f} : E \to \mathbb{R}^2$ be given by $\mathbf{f}(x,y) = (xy, x^2 + y^2)$. Prove $\mathbf{f}^{-1}$ exists and is differentiable in some open set $W$ containing $(2,5)$ and compute $D(\mathbf{f}^{-1})(2,5)$.

    HINT: There are two different answers because two different points in $E$ are mapped to $(2,5)$. To see this, sketch the graphs of $xy = 2$ and $x^2 + y^2 = 5$ in $E$.

**Math 593**

Basic reference: Real Analysis, Second Edition, by Gerald B. Folland

- $\sigma$-algebras. Borel $\sigma$-algebra and generating sets for $\mathbb{R}$, product $\sigma$-algebra, elementary families, §§1.1–1.2.

  Measures. Definition, finite and countable additivity, properties (monotonicity, subadditivity, continuity from above/below), complete measures and completion of measures, $\sigma$-finite sets, §1.3.

  Construction of measures. Outer measures, Carathéodory's Theorem, §1.4.

  Borel measures on $\mathbb{R}$. Distribution functions, Lebesgue-Stieltjes measures, Lebesgue measure §1.5.

- Measurable functions. Definition, equivalent characterizations, properties (sums and products of measurable functions are measurable, $\sup$, $\inf$, $\limsup$, $\liminf$ of measurable functions are measurable); simple functions, approximation by simple functions; the notion of "almost everywhere", §2.1.

  Integration. Integration of nonnegative functions: definition, basic properties, Monotone Convergence Theorem, Series Monotone Convergence Theorem, Fatou's Lemma; itegration of complex functions: definition, basic properties, convergence in $L^1$, Dominated Convergence Theorem, Series Dominated Convergence Theorem, §2.3.

Modes of Convergence. Pointwise convergence, almost everywhere convergence, $L^1$ convergence, Cauchy in measure, convergence in measure; their relationships to one another, §2.4.

Product measures. Rectangles, definition of product measure, sections of sets, sections of functions, monotone class, Monotone Class Theorem, measurability and sections, the Fubini-Tonelli Theorem and the Fubini-Tonelli Theorem for Complete Measures, §2.5.

- $L^p$ spaces. Hölder's Inequality, Minkowski's Inequality; $L^p$ is Banach for $1 \le p < \infty$ and the set of simple functions is dense; definition of $L^\infty$ and the essential supremum; the properties of $L^p$ for $1 \le p < \infty$ extend to $L^\infty$; sequence spaces $\ell^p$; relations between $L^p$ and $L^q$ for $q \ne p$, §6.1.
- Some hints on checking measurability of real valued functions. Let $\mathcal{L}$ denote the Lebesgue measurable sets in $\mathbb{R}$.
  (a) The product, difference and sum of measurable functions is measurable.
  (b) A continuous function from $A \subseteq \mathbb{R} \to \mathbb{R}$ is Borel measurable, hence $\mathcal{L}$ measurable—remember $\mathcal{L}$ is the collection of Lebesgue measurable sets. Reason: $f^{-1}(\text{Borel}) = \text{Borel} \in \mathcal{L}$.
  (c) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then regarded as the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = f(x)$, it is $\mathcal{L}^2$ measurable: $F^{-1}(\text{Borel}) = \{(x, y) : F(x, y) \in \text{Borel}\} = \{(x, y) : f(x) \in \text{Borel}\} = f^{-1}(B) \times \mathbb{R}$, which is measurable.
  (d) Example: Say $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable. Then
  $$G = \{(x, y) : f(x) < y^3\}$$
  is measurable for the product $\sigma$-algebra $\mathcal{L}^2$. To see why, observe that for $h(x, y) = f(x) - y^3$, we have $G = h^{-1}((-\infty, 0))$. Thus it suffices to show $h$ is measurable for $\mathcal{L}^2$. But the function $y \to y^3$ is continuous, hence Lebesgue measurable, hence $\mathcal{L}^2$ measurable by (c). Since $f$ is $\mathcal{L}^2$ measurable by (c) too, it follows that $h$ is $\mathcal{L}^2$ measurable measurable because it is a difference of measurable functions.

Examples of typical homework problems—this is *not* an exhaustive list:

1. Let $X$ be uncountable and define
   $$\mathcal{A} = \{E \subseteq X : \ E \text{ is countable or } E^c \text{ is countable}\}.$$
   Prove $\mathcal{A}$ is a $\sigma$-algebra.
2. Prove that a nonempty collection of subsets of a nonvoid set $X$ is a $\sigma$-algebra iff it is closed under complements and countable intersections.
3. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{E} = \{\{6\}, \{2, 4\}\}$. Find the $\sigma$-algebra generated by $\mathcal{E}$.
4. Let $\mathcal{B}_1, \mathcal{B}_2, \ldots,$ be a countable collection of $\sigma$-algebras. Then $\displaystyle\bigcup_{n=1}^{\infty} \mathcal{B}_n$ need not be a $\sigma$-algebra. In fact, $\mathcal{B}_1 \bigcup \mathcal{B}_2$ need not be a $\sigma$-algebra. Prove the latter, hence the former.

5. Suppose $X \neq \emptyset$ and $\mathcal{E}$ is the set of all one point subsets of $X$. Prove

$$\mathcal{M}(\mathcal{E}) = \{A \subseteq X : \ A \text{ is countable}\} \cup \{A \subseteq X : \ A^c \text{ is countable}\}.$$

6. If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

7. Let $\mu$ be a finitely additive measure on a measurable space $(X, \mathcal{M})$. Prove $\mu$ is countably additive iff it is continuous from below.

8. Let $X$ be countably infinite and let $\mathcal{M} = \mathcal{P}(X)$. Define $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(E) = 0$ if $E$ is finite and $\infty$ if $E$ is infinite.

   (a) Show $\mu$ is finitely additive but not countably additive.

   (b) Show that $X$ is the limit of an increasing sequence of sets $E_n$ with $\mu(E_n) = 0$ for all $n$, but $\mu(X) = \infty$.

9. Using the characterization of Lebesgue measure $m$ on $\mathbb{R}$ as $m((a, b]) = b - a$, prove $a, b \in \mathbb{R}$,

   (a) $m((a, b)) = b - a$

   (b) $m([a, b]) = b - a$

10. Prove that for any Lebesgue measurable set $E \subseteq \mathbb{R}$

$$m(E) = \inf \left\{ \sum_{j=1}^{\infty} m([a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} [a_j, b_j] \right\}.$$

11. Recall the definition of the symmetric difference of sets $A$ and $B$:

$$A \Delta B = (A \backslash B) \cup (B \backslash A).$$

   Let $m$ be Lebesgue measure on $\mathbb{R}$ and suppose $E \subseteq \mathbb{R}$ is Lebesgue measurable w$m(E) < \infty$. Prove that for each $\varepsilon > 0$ there exists a finite union $A$ of open intervals such that $m(A \Delta E) < \varepsilon$.

   HINT: use outer regularity with $\varepsilon/2$. What is the characterization of open subsets in $\mathbb{R}$ in terms of open intervals?

12. Recall that a function $F : \mathbb{R} \to \mathbb{R}$ is *right continuous* if

$$\lim_{y \downarrow x} F(y) = F(x) \quad \text{for all } x \in \mathbb{R}.$$

   Let $\mu$ be a Borel measure on $\mathbb{R}$; that is, $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ is a measure space. Define the *distribution function* $F : \mathbb{R} \to \mathbb{R}$ of $\mu$ by $F(x) = \mu((-\infty, x])$. Since $\mu$ is continuous from above, $F$ is right continuous. By monotonicity of $\mu$, $F$ is nondecreasing. Prove the following:

   (a) For $a < b$, $\mu((a, b]) = F(b) - F(a)$

(b) $\mu(\{a\}) = F(a) - F(a-)$.

(c) $\mu((a,b)) = F(b-) - F(a)$.

(d) $\mu([a,b)) = F(b-) - F(a-)$.

13. Let $f : X \to Y$ and $g : Y \to Z$. Prove $(g \circ f)^{-1} = f^{-1} \circ g^{-1} : \mathcal{P}(Z) \to \mathcal{P}(X)$.

14. Prove $f^{-1}(E^c) = (f^{-1}(E))^c$.

15. Prove that if $\mathcal{M}$ is a $\sigma$-algebra, then so is $f^{-1}(\mathcal{M})$.

16. Given a measurable space $(X, \mathcal{M})$, prove the following are equivalent:

    (a) $f : X \to \overline{\mathbb{R}}$ is measurable.

    (b) $f^{-1}((\lambda, \infty]) \in \mathcal{M}$ for all $\lambda \in \overline{\mathbb{R}}$.

    (c) $f^{-1}([-\infty, \lambda)) \in \mathcal{M}$ for all $\lambda \in \overline{\mathbb{R}}$.

17. For this problem you need the following definition. Given a measurable space $(X, \mathcal{M})$ and $Y \in \mathcal{M}$, we say a function $f : Y \to \overline{\mathbb{R}}$ is measurable on $Y$ if for all $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, $f^{-1}(B) \cap Y \in \mathcal{M}$. This is equivalent to saying $f|_Y$ is $\mathcal{M}_Y$ measurable, where $\mathcal{M}_Y = \{F \cap Y : F \in \mathcal{M}\}$.

    Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then $f$ is measurable iff $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$ and $f$ is measurable on $Y$.

18. If $\{f_n\}$ is a sequence of measurable functions on $X$, then $\{x : \lim f_n(x) \text{ exists }\}$ is a measurable set.

    HINT: $\{x : f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} \{x : f(x) < r \leq g(x)\}$

19. If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then $f$ is measurable.

20. If $X = A \cup B$, where $A, B \in \mathcal{M}$, then $f : X \to \mathbb{R}$ is measurable iff $f$ is measurable on $A$ and on $B$.

21. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, the $f$ is Borel measurable.

    HINTS:

    - Explain why $f$ measurable implies $-f$ measurable and use this fact to show it suffices to consider $f$ monotone nondecreasing.
    - Prove $f^{-1}([a, \infty))$ is an interval: recall an interval in $\mathbb{R}$ is any set $I$ such that $x, y \in I$ and $z$ between $x$ and $y$ implies $x \in I$.

22. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $f$ is measurable and nonnegative. Then $\lambda(E) = \int_E f \, d\mu$ is a measure on $\mathcal{M}$

23. Suppose $\{f_n\} \subseteq L^1(\mu)$ and $f_n \to f$ uniformly. If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.

24. On $([0,1], \mathcal{B}_{[0,1]}, m)$, let $f_n = n^{3/2} I_{(1/(n+1),1/n)}$. Prove DCT does not apply because the domination condition fails. Then prove $f_n \to 0$ everywhere and $\int f_n \to 0$.

    HINT: Sketch the $f_n$ on the same graph and see what $g \geq f_n$ means.

25. Suppose $f \in L^1(m)$, where $m$ is Lebesgue measure on $\mathbb{R}$, and $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\, dm(t)$. Prove $F$ is continuous on $\mathbb{R}$.

26. If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

27. If $f_n \to f$ in measure and $g_n \to g$ in measure, then $f_n + g_n \to f + g$ in measure.

28. Let $\mu$ be counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

29. Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, and $\mu = \nu =$ counting measure. Define

$$f(m,n) = \begin{cases} 1, & m = n \\ -1, & m = n+1 \\ 0, & \text{otherwise.} \end{cases}$$

    Prove that $\int |f| d(\mu \times \nu) = \infty$ while $\iint f\, d\mu\, d\nu$ and $\iint f\, d\nu\, d\mu$ exist and are unequal.

    HINTS: Note that $\mu \times \nu$ is a counting measure on the product space—this is easy to verify (do it). There is a measurable set $E$ for which $|f| = I_E$.

30. If $f$ is Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a t^{-1} f(t)\, dt$, then $g$ is integrable on $(0,a)$ and $\int_0^a g(x)\, dx = \int_0^a f(x)\, dx$. Make sure you verify the hypotheses of any theorem you use.

    HINT: Write out $\int_0^a g(x)\, dx$ explicitly as a double integral $\int_0^a \int_0^a h(x,t)\, dt, dx$. Use Tonelli to show $|h|$ is in $L^1$ then use Fubini to finish.

31. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(Y, \mathcal{N}, \nu) = ([0,\infty), \mathcal{B}_{[0,\infty)}, m)$, where $m$ is Lebesgue measure. Use the Fubini-Tonelli Theorem to prove that if $f : X \to [0,\infty)$ is measurable, then

$$\int f\, d\mu = \int_0^\infty \mu(\{x : f(x) > y)\}\, dy,$$

    where we follow the standard convention to write $dy$ for $dm(y)$. Make sure you verify the hypotheses of any theorem you use.

32. On the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$ (where $m$ is Lebesgue measure), let

$$f_n(x) = n(1 - nx) I_{[0,1/n]}(x) - \frac{1}{1+x^2} I_{(1/n,\infty)}(x).$$

    (a) Show $\{f_n\}$ converges a.e. and identify the limit $f$.
    (b) Does $\int |f_n - f| \to 0$?
    (c) Does $f_n \to f$ in measure?

    It might help you to find the negative and positive parts of $f - f_n$.