Basic Real Analysis Comprehensive Exam Syllabus

Math 528

Basic references: Rudin's Principles of Mathematical Analysis and Wade's An Introduction to Analysis, Fourth Edition.

- Rudin: Chapter 2; Chapter 3: 3.1–3.19, 3.38; Chapter 4: 4.8, 4.13–4.23; Chapter 5: 5.15; Chapter 7; Chapter 9: 9.10–9.21, 9.24–9.28. There is a gentler approach to chapter 9 in Wade's book—see the next bullet point.
- Wade: Chapter 11: Section 11.1 through Example 11.3; Section 11.2; Section 11.4; Section 11.6

Examples of typical homework problems (not an exhaustive list)

THIS IS NOT A LIST OF POTENTIAL EXAM QUESTIONS

1. Rudin page 44/11
2. Let $C[a,b]$ be the set of continuous functions on the interval $[a,b]$. For $f,g \in C[a,b]$ define
$$d(f,g) = \int_a^b |f(x) - g(x)|\,dx.$$
Prove this is a metric on $C[a,b]$.
3. Rudin page 43/5
4. Rudin page 43/9
5. Determine the union and prove your answer:
$$\bigcup_{n=3}^{\infty} \left[1 + \frac{1}{n}, 2 - \frac{2}{n}\right).$$
6. A point x in a metric space X is a *boundary point* of $E \subseteq X$ if for each $\varepsilon > 0$,
$$N_\varepsilon(x) \cap E \neq \emptyset \qquad \text{and} \qquad N_\varepsilon(x) \cap E^c \neq \emptyset.$$
The set of all boundary points of E is called the *boundary* of E and is denoted by ∂E.
   (a) Prove E is closed iff $\partial E \subseteq E$.
   (b) Prove $E \cup \partial E = \overline{E}$.
   (c) Show there are sets $A, B \subseteq \mathbb{R}$ such that $\partial(A \cup B) \neq (\partial A) \cup (\partial B)$.
   (d) Show there are sets $A, B \subseteq \mathbb{R}$ such that $\partial(A \cap B) \neq (\partial A) \cap (\partial B)$.
   (e) Prove $\partial E = \overline{E} \backslash E^o$.
7. Let X be a metric space and $E \subseteq Y \subseteq X$. Show E is closed relative to Y iff for some closed $F \subseteq X$, $E = Y \cap F$.

8. Suppose X is a metric space and $A, B \subseteq X$ are compact. Prove $A \cap B$ and $A \cup B$ are compact.

9. Prove that $\partial(A \cap B) \cap (A^c \cup (\partial B)^c) \subseteq \partial A$.

10. Prove that if $\{p_n\}$ is Cauchy in the metric space X and some subsequence converges, then $\{p_n\}$ converges.

11. Equip $\mathbb{R}$ with the discrete metric. Prove the resulting metric space is complete.

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x + y$. Sketch $f^{-1}([0, 1])$. HINT: what is $f^{-1}(\{c\})$?

13. Prove $f^{-1}(E^c) = (f^{-1}(E))^c$.

14. Rudin page 98/2

15. Rudin page 98/3

16. A metric space X is *separable* if it contains a countable dense subset.

    A separable metric space has the Lindelöf Property: If $\{V_\alpha\}$ is an open cover of $E \subseteq X$, then there are countably many $\alpha_1, \alpha_2, \ldots$ such that

    $$E \subseteq \bigcup_{n=1}^{\infty} V_{\alpha_n}.$$

    This was proved in 527 for $\mathbb{R}$ using that $\mathbb{Q}$ is dense in $\mathbb{R}$—the same proof works for any separable metric space.

    In a metric space X, let $\{V_\alpha\}_{\alpha \in A}$ be a collection of nonempty open sets satisfying $V_\alpha \cap V_\beta = \emptyset$ for all $\alpha \neq \beta$ in A. Prove that if X is separable, then A is countable.

17. Suppose $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$. Prove the following identity for the inverse images:

    $$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

18. Use the previous problem to give an EASY proof of the composition rule: Let X, Y,and Z be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions then $g \circ f : X \to Z$ is also continuous.

19. Rudin page 116/17 HINT to the hint in the text. After following the hint in the text, assume $f^{(3)}(x) < 3$ for all $x \in (-1, 1)$ and use the last equation in the hint to get a contradiction.

20. Prove for $x \in (0, \pi)$ and $n \in \mathbb{N}$,

    $$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - \frac{x^{4n-1}}{(4n-1)!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{4n+1}}{(4n+1)!}$$

21. Rudin page 166/5, first part only (i.e., omit the last sentence in the problem)

22. Evaluate the limit
$$\lim_{n\to\infty}\int_0^3 \sqrt{x+1+\sin\frac{x}{n}}\,dx$$
and justify your answer.

23. Rudin page 165/7

24. Rudin page 165/9, omit the part about the converse

25. Rudin page 165/1

26. Rudin page 165/2 HINT: the previous problem will be useful.

27. Suppose K is a compact metric space and E is a countable dense subset of K. Given $\delta > 0$, prove there are $x_1, \ldots, x_n \in E$ such that
$$K \subseteq N_\delta(x_1) \cup \cdots \cup N_\delta(x_n).$$

28. Rudin page 79/10

29. Rudin page 82/23

30. Suppose $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R \in (0, \infty)$.

    (a) Find the radius of convergence of $\sum_{k=0}^{\infty} a_k x^{2k}$.

    (b) Find the radius of convergence of $\sum_{k=0}^{\infty} a_k^2 x^k$.

31. Suppose $\{a_k\}$ is a bounded sequence of real numbers. Prove $\sum_{k=0}^{\infty} a_k x^k$ has a positive radius of convergence.

32. Prove that
$$f(x) = \sum_{k=0}^{\infty} \left(\frac{x}{(-1)^k + 4}\right)^k$$
is differentiable on $(-3, 3)$.

33. Rudin page 168/18

    Useful facts: Theorem 6.20 in the text, and the fact that if f is Riemann integrable on on $[a, b]$, then so is $|f|$ and $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$.

34. Rudin page 169/20

35. Rudin page 168/16

    HINT: Get δ from equicontinuity for $\varepsilon/3$ and cover K with $\{N_\delta(y)\}_{y\in K}$.

36. Decide if the limit exists. If it does, find it. Justify your answer.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{(x^2+y^2)^{1/3}}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^2+2y^4}$.

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x \sin y}{x^2+y^2}$.

37. Compute f_x and find where it is continuous.

(a) $f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{(x^2+y^2)^{1/3}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

38. Rudin page 239/6 (Note: $D_1 f = f_x$ and $D_2 f = f_y$).

39. Let

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Prove f is differentiable, but f' is not continuous.

40. Prove that the first partials of $f(x,y) = (xy)^{1/5}$ exist at $(x,y) = (0,0)$, but f is not differentiable there.

41. Let $U \subset \mathbb{R}^n$ be open. Suppose $f : U \to \mathbb{R}$ is differentiable and positive. Prove

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}.$$

42. Let $u(x,t) = \dfrac{e^{-x^2/4t}}{\sqrt{4\pi t}}$, $t > 0$, $x \in \mathbb{R}$. If $a > 0$, show $u(x,t) \to 0$ as $t \to 0^+$, uniformly for $x \in [a, \infty)$.

43. Let $u : \mathbb{R} \to [0,\infty)$ be differentiable and set $F(x,y,z) = u(\sqrt{x^2+y^2+z^2})$. For $(x,y,z) \neq (0,0,0)$, compute

$$\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}.$$

44. Rudin page 239/8 HINT: f has a local maximum at $\mathbf{x} \in E$—where $E \subseteq \mathbb{R}^n$ is open—if there exists $\delta > 0$ such that $f(\mathbf{y}) \leq f(\mathbf{x})$ for all $\mathbf{y} \in N_\delta(\mathbf{x})$. Compute $\frac{\partial f}{\partial x_j}(\mathbf{x})$.

45. Text page 240/13

Math 593

Basic reference: Folland's Real Analysis

- Chapter 1; Chapter 2: 2.1–2.6; Chapter 6: 6.1.

Here are some typical homework problems.

1. Let X be uncountable and define
$$\mathcal{A} = \{E \subseteq X : \ E \text{ is countable or } E^c \text{ is countable}\}.$$
Prove $\mathcal{A}$ is a σ-algebra.

2. Prove that a nonempty collection of subsets of a nonvoid set X is a σ-algebra iff it is closed under complements and countable intersections.

3. (a) Using parts (a)–(b) of Proposition 1.2, prove the part about $\mathcal{E}_3$ in (c).
(b) Using parts (a)–(c) of Proposition 1.2, prove the part about $\mathcal{E}_5$ in (d).

4. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{E} = \{\{6\}, \{2, 4\}\}$. Find the σ-algebra generated by $\mathcal{E}$.

5. Let $\mathcal{B}_1, \mathcal{B}_2, \ldots,$ be a countable collection of σ-algebras. Then $\bigcup\limits_{n=1}^{\infty} \mathcal{B}_n$ need not be a σ-algebra. In fact, $\mathcal{B}_1 \bigcup \mathcal{B}_2$ need not be a σ-algebra. Prove the latter, hence the former.

6. Suppose $X \neq \emptyset$ and $\mathcal{E}$ is the set of all one point subsets of X. Prove
$$\mathcal{M}(\mathcal{E}) = \{A \subseteq X : \ A \text{ is countable}\} \cup \{A \subseteq X : \ A^c \text{ is countable}\}.$$

7. Text §1.3, page 27/6

8. Text §1.3, page 27/9

9. Let μ be a finitely additive measure on a measurable space $(X, \mathcal{M})$. Prove μ is countably additive iff it is continuous from below.

10. Let X be countably infinite and let $\mathcal{M} = \mathcal{P}(X)$. Define $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(E) = 0$ if E is finite and ∞ if E is infinite.
    (a) Show μ is finitely additive but not countably additive.
    (b) Show that X is the limit of an increasing sequence of sets E_n with $\mu(E_n) = 0$ for all n, but $\mu(X) = \infty$.

11. Text §1.4/17

12. Text §1.4/18ab HINT on (b), use part (a)

13. Text §1.4/19 HINT: On $\Leftarrow$ use 18(a) with $\varepsilon = 1/n$ to get corresponding A_n and use what it means for A_n to be μ^* measurable.

14. For $a, b \in \mathbb{R}$, prove for Lebesgue measure m,

    (a) $m((a,b)) = b - a$

    (b) $m([a,b]) = b - a$

15. Prove that for any Lebesgue measurable set $E \subseteq \mathbb{R}$

$$m(E) = \inf \left\{ \sum_{j=1}^{\infty} m([a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} [a_j, b_j] \right\}.$$

16. Recall the definition of the symmetric difference of sets A and B:

$$A \Delta B = (A \backslash B) \cup (B \backslash A).$$

Let m be Lebesgue measure on $\mathbb{R}$ and suppose $E \subseteq \mathbb{R}$ is Lebesgue measurable w$m(E) < \infty$. Prove that for each $\varepsilon > 0$ there exists a finite union A of open intervals such that $m(A \Delta E) < \varepsilon$.

HINT: use outer regularity with $\varepsilon/2$. What is the characterization of open subsets in $\mathbb{R}$ in terms of open intervals?

17. Recall that a function $F : \mathbb{R} \to \mathbb{R}$ is *right continuous* if

$$\lim_{y \downarrow x} F(y) = F(x) \quad \text{for all } x \in \mathbb{R}.$$

Let μ be a Borel measure on $\mathbb{R}$; that is, $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ is a measure space. Define the *distribution function* $F : \mathbb{R} \to \mathbb{R}$ of μ by $F(x) = \mu((-\infty, x])$. Since μ is continuous from above, F is right continuous. By monotonicity of μ, F is nondecreasing. Prove the following:

    (a) For $a < b$, $\mu((a,b]) = F(b) - F(a)$

    (b) $\mu(\{a\}) = F(a) - F(a-)$.

    (c) $\mu((a,b)) = F(b-) - F(a)$.

    (d) $\mu([a,b)) = F(b-) - F(a-)$.

18. Let $f : X \to Y$ and $g : Y \to Z$. Prove $(g \circ f)^{-1} = f^{-1} \circ g^{-1} : \mathcal{P}(Z) \to \mathcal{P}(X)$.

19. Prove $f^{-1}(E^c) = (f^{-1}(E))^c$.

20. Prove that if $\mathcal{M}$ is a σ-algebra, then so is $f^{-1}(\mathcal{M})$.

21. Given a measurable space $(X, \mathcal{M})$, prove the following are equivalent:

    (a) $f : X \to \overline{\mathbb{R}}$ is measurable.

    (b) $f^{-1}((\lambda, \infty]) \in \mathcal{M}$ for all $\lambda \in \overline{\mathbb{R}}$.

    (c) $f^{-1}([-\infty, \lambda)) \in \mathcal{M}$ for all $\lambda \in \overline{\mathbb{R}}$.

22. Let $(X, \mathcal{M})$ be a measurable space and suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable. With the convention that $\infty - \infty = 0$, prove $h = f + g$ is measurable.

    HINTS:

    (a) Explain why $E_\infty = \{x \in X : f(x) = -g(x) = \pm\infty\}$ is measurable.
    (b) Find $h^{-1}(\{\infty\})$ and $h^{-1}(\{-\infty\})$ in terms of f^{-1} and g^{-1}.
    (c) Look at $h^{-1}((b, \infty))$ for $b \in \mathbb{R}$ and consider cases $0 \leq b$ and $b < 0$.

23. Text page 48/1.

    For this problem you need the following definition. Given a measurable space $(X, \mathcal{M})$ and $Y \in \mathcal{M}$, we say a function $f : Y \to \overline{\mathbb{R}}$ is measurable on Y if for all $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, $f^{-1}(B) \cap Y \in \mathcal{M}$. This is equivalent to saying $f|_Y$ is $\mathcal{M}_Y$ measurable, where $\mathcal{M}_Y = \{F \cap Y : F \in \mathcal{M}\}$.

24. Text page 48/3

    HINT: $\{x : f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} \{x : f(x) < r \leq g(x)\}$

25. Text page 48/4

26. Text page 48/5

27. Text page 48/8

    HINTS:

    - Explain why f measurable implies $-f$ measurable and use this fact to show it suffices to consider f monotone nondecreasing.
    - Prove $f^{-1}([a, \infty))$ is an interval: recall an interval in $\mathbb{R}$ is any set I such that $x, y \in I$ and z between x and y implies $x \in I$.

28. Text page 52/13.

    HINTS: You want to show $\limsup \int_E f_n \leq \int_E f \leq \liminf \int_E f_n$. To get the lower inequality, look at $\int_E f = \int f - \int_{E^c} f$.

    You might find the following properties of $\limsup$ and $\liminf$ useful:

    (a) $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$
    (b) $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$
    (c) $-\limsup a_n = \liminf(-a_n)$
    (d) $-\liminf a_n = \limsup(-a_n)$
    (e) If $\lim a_n$ exists then $\liminf(a_n + b_n) = \lim a_n + \liminf b_n$ and $\limsup(a_n + b_n) = \lim a_n + \limsup b_n$

    For the second part, write $f_n = f + g_n$ where f and g_n are nonzero on disjoint sets.

29. Text page 52/14

30. Text page 58/18

31. Text page 58/19

32. Text page 58/20

33. Text page 58/21

34. Text page 60/26

35. Text page 63/33

36. Text page 63/38

37. Text page 63/39

38. Text page 63/42

    Some hints on checking measurability of real valued functions:

    (a) The product, difference and sum of measurable functions is measurable (Prop. 4.6)

    (b) A continuous function from $A \subseteq \mathbb{R} \to \mathbb{R}$ is Borel measurable, hence $\mathcal{L}$ measurable—remember $\mathcal{L}$ is the collection of Lebesgue measurable sets. Reason: $f^{-1}(\text{Borel}) = \text{Borel} \in \mathcal{L}$.

    (c) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then regarded as the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = f(x)$, it is $\mathcal{L}^2$ measurable: $F^{-1}(\text{Borel}) = \{(x, y) : F(x, y) \in \text{Borel}\} = \{(x, y) : f(x) \in \text{Borel}\} = f^{-1}(B) \times \mathbb{R}$, which is measurable.

    (d) Example: Say $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable. Then

    $$G = \{(x, y) : f(x) < y^3\}$$

    is measurable for the product σ-algebra $\mathcal{L}^2$. To see why, observe that for $h(x, y) = f(x) - y^3$, we have $G = h^{-1}((-\infty, 0))$. Thus it suffices to show h is measurable for $\mathcal{L}^2$. But the function $y \to y^3$ is continuous, hence Lebesgue measurable, hence $\mathcal{L}^2$ measurable by (c). Since f is $\mathcal{L}^2$ measurable by (c) too, it follows that h is $\mathcal{L}^2$ measurable measurable because it is a difference of measurable functions.

39. Text page 68/46

    HINT on evaluating $\iint I_D d(\mu \times \nu)$: Use the definition of $\mu \times \nu$ as outer measure and note that for a rectangle $A \times B$ with $\mu(A \cap B) > 0$, it must be true that $A \cap B$ is infinite (explain why).

40. Text page 69/48

    HINTS: Note that $\mu \times \nu$ is a counting measure on the product space—this is easy to verify (do it). There is a measurable set E for which $|f| = I_E$.

41. Page 77/56. Make sure you verify the hypotheses of any theorem you use.

    HINT: Write out $\int_0^a g(x)\,dx$ explicitly as a double integral $\int_0^a \int_0^a h(x,t)\,dt, dx$. Use Tonelli to show $|h|$ is in L^1 then use Fubini to finish.

42. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(Y, \mathcal{N}, \nu) = ([0,\infty), \mathcal{B}_{[0,\infty)}, m)$, where m is Lebesgue measure. Use the Fubini-Tonelli Theorem to prove that if $f : X \to [0,\infty)$ is measurable, then

    $$\int f\,d\mu = \int_0^\infty \mu(\{x : f(x) > y)\}\,dy,$$

    where we follow the standard convention to write dy for $dm(y)$. Make sure you verify the hypotheses of any theorem you use.

43. If $1 \leq p < r \leq \infty$, prove that $L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$. Make sure you verify that the set is a vector space and $\|f\|$ is a norm.