Math 525 Masters Exam Syllabus
Text: Linear Algebra Done Right, Third Edition, by Sheldon Axler
Chapters: 1-3, 5.
Examples of typical homework problems (not an exhaustive list):

Throughout $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and U, V, W are vector spaces over $\mathbb{F}$.

1. Let U_1, U_2 be subspaces of V. Prove that $U_1 \cap U_2$ is a subspace of V.
2. Let U_1, U_2 be subspaces of V. Prove that $U_1 \cup U_2$ is a subspace of V if and only if either $U_1 \subset U_2$ or $U_2 \subset U_1$.
3. Prove or give a counterexample: If U_1, U_2, W are subspaces such that $V = U_1 \oplus W$ and $V = U_2 \oplus W$, then $U_1 = U_2$.
4. Let U be the set of all polynomial functions f of the form $f(x) = a + bx + cx^2$, where $a, b, c \in \mathbb{F}$. Prove that U is a subspace of $\mathbb{F}[x]$.
5. Prove that if $v_1, \ldots, v_n$ are linearly independent in V, then so is the list $v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n$.
6. Let $W = \{(a, b, c) \in \mathbb{R}^3 \mid a + 2b + 3c = 0\}$. Prove that W is finite-dimensional by finding a finite spanning set for W.
7. Suppose that V is finite-dimensional and U is a subspace of V such that $\dim V = \dim U$. Prove that $U = V$.
8. Prove or disprove: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3, v_4 \notin U$, then v_1, v_2 is a basis of U.
9. Suppose that V is finite dimensional. Prove that if $U_1, \ldots, U_m$ are subspaces of V such that $V = U_1 \oplus \ldots \oplus U_m$, then $\dim V = \dim U_1 + \ldots + \dim U_m$.
10. Suppose that V is finite-dimensional. If U, W are subspaces with $\dim U + \dim W > \dim V$, prove that $U \cap W \neq \{0\}$.
11. Suppose that $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $T(v_1), \ldots, T(v_m)$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.
12. Suppose that $\dim V = 1$. Show that for any $T \in \mathcal{L}(V, V)$ there exists $\lambda \in \mathbb{F}$ such that $T(v) = \lambda v$ for all $v \in V$.
13. Suppose V is finite-dimensional and let U be a subspace of V and $S \in \mathcal{L}(U, W)$. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $T(u) = S(u)$ for all $u \in U$.
14. Let $S, T \in \mathcal{L}(V, V)$ such that range $S \subset$ null T. Prove that $(ST)^2 = 0$.
15. Suppose that T is a linear map from V to $\mathbb{F}$. Prove that if $u \in V$ is not in null T, then $V = \text{null } T \oplus \{au : a \in F\}$.
16. Prove that if $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ are linearly independent in V, then $T(v_1), \ldots, T(v_n)$ are linearly independent in W.
17. Prove that if $T \in \mathcal{L}(V, W)$ is surjective and $v_1, \ldots, v_n$ span V, then $T(v_1), \ldots, T(v_n)$ span W.
18. Suppose that $T \in \mathcal{L}(\mathbb{F}^5, \mathbb{F}^3)$ such that null $T = \{(x_1, \ldots, x_5) \in \mathbb{F}^5 \mid x_1 + x_2 = 0 \text{ and } x_4 = 3x_5\}$. Prove that T is not surjective.
19. Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null S and range S are invariant under T.

20. Suppose that $T, S \in \mathcal{L}(V)$ and that S is invertible.
(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.
(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

21. Suppose that $T \in \mathcal{L}(V)$ is invertible and $\lambda \in \mathbb{F} \setminus \{0\}$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

22. Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

23. Suppose that $S, T \in \mathcal{L}(V)$ and S is invertible. Let $f(x) \in \mathbb{F}[x]$ be a polynomial. Prove that $f(STS^{-1}) = Sf(T)S^{-1}$.

24. Suppose that $T \in \mathcal{L}(V)$ and U is a subspace of V that is invariant under T. Prove that U is invariant under $f(T)$ for all $f(x) \in \mathbb{F}[x]$.

25. Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \text{null } T \oplus \text{range } T$.

26. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

27. Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V and that $\lambda \in \mathbb{F}$. Prove that λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

28. Suppose that $T \in \mathcal{L}(\mathbb{F}^5)$ and $\dim E(8, T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

29. Suppose that $T \in \mathcal{L}(V)$ is invertible. Prove that $E(\lambda, T) = E(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.