MATH 527 Masters Exam Syllabus

Text: Understanding Analysis, Second Edition, by Stephen Abbott

Coverage: Chapters 1–5, 7.

Examples of typical homework problems (not an exhaustive list):

THIS IS NOT A LIST OF POTENTIAL EXAM QUESTIONS

1. Text/1.2.3(c), 1.2.6(d), 1.2.2
2. Prove if a is and integer and a^2 is odd, then a itself is odd.
3. Prove $\sqrt{6}$ is irrational.
4. Prove $\inf(0,1) = 0$.
5. Find the inf and sup of the interval $[-1,2)$; justify your answer.
6. Find the inf and sup of the set $\left\{\frac{1}{n} + n : n \in \mathbb{N}\right\}$; justify your answer.
7. Find the inf and sup of $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$ and justify your answer.
8. Text/1.3.2, 1.3.8, 1.3.11, 1.4.1, 1.4.5
9. Prove the set
$$A = \left\{\frac{2n-1}{3n^5+5} : n \in \mathbb{N}\right\}$$
is countable.
10. A *dyadic rational* number has the form $n/2^m$, where $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Decide if the set of dyadic rationals is countable or uncountable and justify your answer.
11. Given $n \in \mathbb{N}$ with $n \geq 2$, the Cartesian product of the nonempty sets $B_1, B_2, \ldots, B_n$ is the set
$$B_1 \times \cdots \times B_n = \{(b_{1,\ell_1}, b_{2,\ell_2}, \ldots, b_{n,\ell_n}) : b_{k,\ell_k} \in B_k\}.$$
If $B_1, B_2, \ldots, B_n$ are all countable, use induction to prove their Cartesian product is countable. Make sure you spell out your induction clearly.
12. Text 2.2.2, 2.3.6, 2.3.7(b)–(e), 2.3.2 (make sure you use the $\varepsilon - N$ definition and *not* a theorem)
13. Use the definition of the limit to prove the following.

    (a) $\displaystyle\lim_{n\to\infty} \frac{3n^3+2n^2+8}{4n^3+9} = \frac{3}{4}$.

    (b) $\displaystyle\lim_{n\to\infty} \frac{1}{5n^2-1} = 0$.

    (c) $\displaystyle\lim_{n\to\infty} \frac{1}{6n^2-8n+1} = 0$.

14. We say a sequence $a_n \to \infty$ if for each $M > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Longrightarrow a_n > M.$$

   (a) State the corresponding definition for $a_n \to -\infty$.
   (b) Prove that if $x_n \to \infty$, then (x_n) has a minimum; that is, for some $N \in \mathbb{N}$, $x_n \geq x_N$ for all $n \in \mathbb{N}$.

15. Suppose that the sequence (a_n) is increasing and it has a convergent subsequence. Does this imply that (a_n) is convergent? Prove or provide a counterexample.

16. Is there any sequence (a_n) which has no convergent subsequence, but $(|a_n|)$ is convergent? Justify your answer.

17. Prove that if (a_n) is bounded, and all of its convergent subsequences converge to a, then (a_n) also converges to a.

18. Prove or provide a counterexample:

   (a) If (x_n) Cauchy and (y_n) is bounded, then $(x_n y_n)$ is Cauchy.
   (b) If (x_n) and (y_n) are Cauchy and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\dfrac{x_n}{y_n}$ is Cauchy.

19. Give $\varepsilon - \delta$ proofs of the following limits.

   (a) $\lim\limits_{x \to 2} (3x^2 - 5x + 1) = 3$
   (b) $\lim\limits_{x \to 3} \dfrac{x-4}{x-2} = -1$
   (c) $\lim\limits_{x \to 2} \dfrac{x^2+1}{x-3} = -5$
   (d) $\lim\limits_{x \to 1} \dfrac{2x-5}{x-3} = \dfrac{3}{2}$

20. Prove the following limits do not exist.

   (a) $\lim\limits_{x \to -5} \dfrac{|x+5|}{x+5}$
   (b) $\lim\limits_{x \to 1} \sin \dfrac{1}{x-1}$

21. Text 4.2.6bc

22. State the definition of the following.

   (a) $\lim\limits_{x \to -\infty} f(x) = -\infty$
   (b) $\lim\limits_{x \to a} f(x) = -\infty$

23. Determine the limits (using methods you learned in Calculus I) and then use the definition to prove your answer.

(a) $\lim_{x\to\infty} \frac{x^3+2}{1-2x^3}$

(b) $\lim_{x\to-\infty} \frac{x-3}{x+4}$

(c) $\lim_{x\to-\infty} \frac{x^2-1}{x^2-3}$

(d) $\lim_{x\to 4^+} \frac{1}{x^2-9x+20}$

(e) $\lim_{x\to\infty} (x-3x^2)$

(f) $\lim_{x\to\infty} \frac{1}{x^2-x-6}$

24. Text 4.2.6(d), 4.3.6 (b)(c)(d), 4.3.8(b)

25. Give an example of infinite number of closed sets whose union is not closed.

26. Define $A \backslash B = \{a \in A : b \notin B\}$. Prove that if A is closed and B is open, then $A \backslash B$ is closed.

27. Text 3.2.2, 3.2.4, 3.2.7(a), 3.2.8(c)—justify your answer, 3.2.11, 3.2.13, 3.3.4—justify your answers, 3.4.5,

28. Prove that the equation $e^x = 2\cos x + 1$ has at least one solution in $\mathbb{R}$.

29. Prove that if f is continuous on $[a,b]$ with $f(x) > 0$ for all $x \in [a,b]$, then $1/f$ is bounded on $[a,b]$.

30. Give a short justification for your answer to each of the following questions. When applicable, a sketch will suffice.

    (a) Is it possible for a continuous function f defined on $[0,1]$ to satisfy $f([0,1]) = (0,1)$?

    (b) Is it possible for a continuous function f defined on $(0,1)$ to satisfy $f((0,1)) = [0,1]$?

31. Let $f : [0,1] \to [0,1]$ be continuous. Prove that f has at least one fixed point in $[0,1]$; that is, there exists $x \in [0,1]$ such that $f(x) = x$.

    HINT: Consider the function $g(x) = f(x) - x$ and apply the Intermediate Value Theorem.

32. Prove that the function $f(x) = 1/x$ is not uniformly continuous on $(0,1)$.

33. Suppose $f : A \to \mathbb{R}$ is uniformly continuous on A. Prove that if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence.

34. Show $f(x) = 1/x^2$ is uniformly continuous on $[1,\infty)$, but not on $(0,1)$.

35. Text 5.2.5, 5.2.7, 5.3.1(a), 5.3.3, 5.3.8, 5.3.12

36. Let $f(x) = x|x|$. Prove that

$$\lim_{h \to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} = 0$$

but $f''(0)$ does not exist.

37. Text 7.2.2ab, 7.2.3, 7.2.7 (HINT: Use the partition P_n where the points are equally spaced $1/n$ units apart).

38. Suppose $f, g : A \to \mathbb{R}$, where A is *any* nonempty set. For example, A could be a set of real numbers, or A could even be sets of sets (for instance, the set of all partitions of an interval $[a, b]$). Prove that

$$\sup_A (f + g) \leq \sup_A f + \sup_A g \qquad \inf_A (f + g) \geq \inf_A f + \inf_A g \qquad .$$

39. Text 7.5.9