NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 3

Fall 2021

Problem 3.

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function that satisfies the following conditions:

(i) $f(f(n)) = n$ for every n.

(ii) $f(f(n+2)+2) = n$ for every n.

(iii) $f(0) = 1$.

Show that $f(n) = 1 - n$ for every n.

Solution.

From (i) we obtain $f(1) = f(f(0)) = 0$. From (ii) it follows that $f(2) = f(f(1)+2) = -1$. Now, we show the following claim: for every a positive integer $n \geqslant 2$ we have $f(n) = 1 - n$ and $f(2-n) = n-1$. We note that this claim finishes the proof. We proceed by induction.

For $n = 2$ we have $f(2) = -1$ and $f(0) = 1$.

Let $n \geqslant 3$ and assume $f(n-1) = 2-n$ and $f(3-n) = n-2$. From (ii) it follows that $f(n) = f(f((1-n)+2))+2) = 1-n$; and from (i) it follows that $f(2-n) = f(f(n-1)) = n-1$. The proof is now complete.