NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 1

Spring 2021

Problem 1.

Let S_n be the sum of the first n terms of the sequence

$$0, 1, 1, 2, 2, 3, 3, 4, 4, 5, \cdots$$

where the nth term of the sequence is given by

$$a_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Show that if n and m are positive integers and $n > m$ then $nm = S_{n+m} - S_{n-m}$.

Solution.

The first step is to find a formula for S_n. We do this for two different cases, n is odd, and n is even.

<u>n **is odd:**</u> In this case we have that S_n has the following form:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ &= 0 + 1 + 1 + 2 + 2 + \cdots + \frac{n-1}{2} + \frac{n-1}{2} \\ &= 2\left(1 + 2 + 3 + \cdots + \frac{n-1}{2}\right) = \frac{(n-1)(n+1)}{4} = \frac{n^2-1}{4}. \end{aligned}$$

<u>n **is even:**</u> In this case we have, $n-1$ is odd. So, using the previous case we obtain:

$$\begin{aligned} S_n = a_n + S_{n-1} &= \frac{n}{2} + \frac{(n-1)^2-1}{4} \\ &= \frac{n^2}{4}. \end{aligned}$$

We note that S_{n+m} and S_{n-m} are both even or odd. We can use the formulas computed above in these two cases:

<u>S_{n+m} **and** S_{n-m} **are odd:**</u>

$$S_{n+m} - S_{n-m} = \frac{(n+m)^2-1}{4} - \frac{(n-m)^2-1}{4} = nm.$$

S_{n+m} and S_{n-m} are even:

$$S_{n+m} - S_{n-m} = \frac{(n+m)^2}{4} - \frac{(n-m)^2}{4} = nm.$$

The solution is finished.