

NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 5

Spring 2021

Problem 5.

Let Δ be a right triangle in the plane whose sides have integer lengths. Show that Δ is similar to a right triangle whose hypotenuse is on the x -axis and whose vertices have integer coordinates.

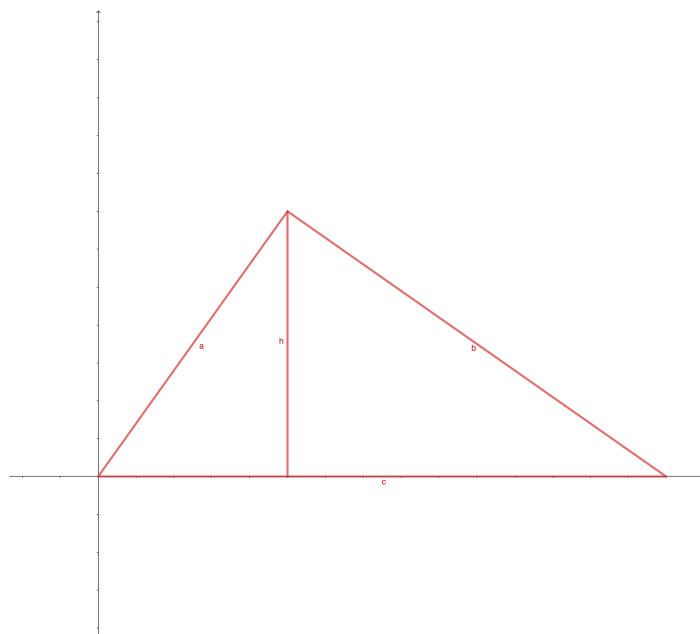
Solution.

Let a, b, c be the lengths of the sides of Δ , where c corresponds to the hypotenuse. Let h be the length of the height H of the triangle dropped from the vertex opposite to the hypotenuse. Then, the area of Δ is:

$$\text{Area}(\Delta) = \frac{c \cdot h}{2} = \frac{a \cdot b}{2}.$$

Thus, we obtain $h = \frac{a \cdot b}{c}$.

We place the hypotenuse of Δ on the positive x -axis so that one of the vertices is at the origin and we obtain the following picture:



The result is a triangle with vertices $(0, 0)$, $(c, 0)$, $(d, \frac{a \cdot b}{2})$. To find the value of d , we note that the triangle to the left of H is similar to Δ . Then, by similarity we obtain:

$$\frac{d}{a} = \frac{a}{c} \quad \Rightarrow \quad d = \frac{a^2}{c}.$$

Now, we dilate Δ with ratio of $2c$ and obtain a triangle similar to Δ whose hypotenuse is on the x -axis and whose vertices are $(0, 0)$, $(2c^2, 0)$, $(2a^2, a \cdot b \cdot c)$, completing the proof.