NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 6

Spring 2021

Problem 6.

Find the result of adding the following infinite series

$$\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{5\cdot 6\cdot 7\cdot 8}+\frac{1}{9\cdot 10\cdot 11\cdot 12}+\cdots$$

Solution.

We show that the answer is

$$\frac{\ln(2)}{4}-\frac{\pi}{24}.$$

Repeatedly using the identity

$$\frac{1}{n(n+a)}=\frac{1}{a}\left(\frac{1}{n}-\frac{1}{n+a}\right).$$

which hold for any pair of natural numbers n and a, we can rewrite the series as

$$\sum_{n=0}^{\infty}\frac{1}{(4n+1)(4n+2)(4n+3)(4n+4)}$$

$$=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{4n+2}-\frac{1}{4n+3}\right)\left(\frac{1}{4n+1}-\frac{1}{4n+4}\right)$$

$$=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{(4n+1)(4n+2)}-\frac{1}{(4n+1)(4n+3)}+\frac{1}{(4n+3)(4n+4)}-\frac{1}{(4n+2)(4n+4)}\right)$$

$$=\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{1}{4n+1}-\frac{3}{4n+2}+\frac{3}{4n+3}-\frac{1}{4n+4}\right).$$

$$=\frac{1}{6}\sum_{n=0}^{\infty}\left(2\left(\frac{1}{4n+1}-\frac{1}{4n+2}+\frac{1}{4n+3}-\frac{1}{4n+4}\right)-\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right)-\left(\frac{1}{4n+1}-\frac{1}{4n+3}\right)\right).$$

Now, evaluating the Maclaurin series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{and} \qquad \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

at $x = 1$, we can rewrite the series as

$$\begin{aligned} \frac{1}{6}\left(2\ln(2) - \frac{1}{2}\ln(2) - \arctan(1)\right) &= \frac{1}{6}\left(2\ln(2) - \frac{1}{2}\ln(2) - \frac{\pi}{4}\right) \\ &= \frac{\ln(2)}{4} - \frac{\pi}{24} \end{aligned}$$

as desired.