

NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 2

Spring 2022

Problem 2.

Find all positive integers n for which the set $\{1, 2, \dots, n\}$ can be split into two disjoint subsets such that the sum of the elements in each subset is the same.

Solution.

Answer: The positive integers of the form $4k$ and $4k - 1$ for k a positive integer.

Since $1 + 2 + \dots + n = \frac{n(n+1)}{2}$, we must have $\frac{n(n+1)}{4}$ is an integer. That is, n is of the form $4k$ or $4k - 1$. To finish the proof we must show that any such n satisfies the condition in the statement. Indeed, this can be observed from the following constructions

$n = 4k$:

$$\{1, \dots, 4k\} = \{1, 4k, 3, 4k - 2, \dots, 2k - 1, 2k + 2\} \cup \{2, 4k - 1, 4, 4k - 3, \dots, 2k, 2k + 1\}.$$

$n = 4k - 1$:

$$\{1, \dots, 4k - 1\} = \{1, 4k - 2, 3, 4k - 4, \dots, 2k - 1, 2k\} \cup \{4k - 1, 2, 4k - 3, 4, 4k - 5, \dots, 2k - 2, 2k + 1\}.$$