NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 6

Spring 2022

Problem 6.

Show that there are only three right triangles with integer sides, up to congruence, whose area is twice the perimeter.

Solution.

Let c be the hypotenuse and a, b the legs of the triangle. Assume without loss of generality that $a \leqslant b$. Then, we have the equations

$$\begin{aligned} c^2 &= a^2 + b^2 \\ \frac{ab}{2} &= 2(a + b + c). \end{aligned}$$

From these equations we obtain $a^2 + b^2 = \left(\frac{ab}{4} - a - b\right)^2$ and then $ab - 8a - 8b + 32 = 0$. After adding 32 at both sides of this equation and factoring the left hand side we obtain

$$(a - 8)(b - 8) = 32.$$

Thus, the pair $(a - 8, b - 8)$ must be equal to one of the pairs $(1, 32)$, $(2, 16)$, $(4, 8)$. From here we obtain the triangles with sides $(9, 40, 41)$, $(10, 24, 26)$, and $(12, 16, 20)$ which satisfy the conditons of the problem.