NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Fall 2023

Problem 8

How many n digit numbers contain an odd number of 1? Express your answer as a simplified formula.

Solution 1: Let A_n and B_n be the number of n-digit numbers that contain an odd and even number of 1 respectively. We have $A_1 = 1$ (since the only 1-digit number with an odd number of 1 is 1 itself), and $B_1 = 8$ (Not counting 0). We first notice that there are $9 \times 10^{n-1}$ n-digit numbers in total. Therefore, $A_n + B_n = 9 \times 10^{n-1}$.

Now consider an $(n+1)$-digit number $\overline{d_1 d_2 \ldots d_{n+1}}$ with an odd number of 1's. There are two cases:

1. If $d_{n+1} = 1$, then $\overline{d_1 d_2 \ldots d_n}$ is an n-digit number with an even number of 1. There are precisely B_n possibilities for $\overline{d_1 d_2 \ldots d_n}$.
2. If $d_{n+1} \neq 1$, then $\overline{d_1 d_2 \ldots d_n}$ is an n-digit number with an odd number of 1. There are precisely A_n possibilities for $\overline{d_1 d_2 \ldots d_n}$ and 9 possibilities for the last digit d_{n+1}.

Therefore, we obtain a recursive formula

$$A_{n+1} = 9A_n + B_n = 9A_n + (9 \times 10^{n-1} - A_n) = 8A_n + 9 \times 10^{n-1}$$

Therefore, we can repeatedly apply this formula to obtain

$$\begin{aligned} A_n &= 8A_{n-1} + 9 \times 10^{n-2} \\ &= 8(A_{n-2} + 9 \times 10^{n-3}) + 9 \times 10^{n-2} = 8^2 A_{n-2} + 9 \times (10^{n-1} + 8 \cdot 10^{n-2}) \\ &= 8^2(8A_{n-3} + 9 \times 10^{n-4}) + 9 \times (10^{n-1} + 8 \cdot 10^{n-2}) \\ &= \ldots \\ &= 8^{n-1} A_1 + 9 \times \left(10^{n-2} + 8 \cdot 10^{n-3} + \cdots + 8^{n-2} \cdot 1\right) \end{aligned}$$

We know $A_1 = 1$, and we can compute the summation using a geometric sum:

$$
\begin{aligned}
10^{n-2} + 8 \cdot 10^{n-3} + \cdots + 8^{n-2} \cdot 1 &= 10^{n-2} \cdot \left(1 + \frac{8}{10} + \cdots + \left(\frac{8}{10}\right)^{n-2}\right) \\
&= 10^{n-2} \frac{1 - \left(\frac{8}{10}\right)^{n-1}}{1 - \frac{8}{10}} \\
&= \frac{9}{2} \cdot \left(10^{n-1} - 8^{n-1}\right).
\end{aligned}
$$

Combined together, we have the formula

$$
A_n = \frac{9}{2} \cdot 10^{n-1} - \frac{7}{2} \cdot 8^{n-1}
$$

Solution 2: Consider an n-digit number $\overline{d_1 d_2 \ldots d_n}$. We divide it into two cases:

1. If $d_1 \neq 1$, we have 8 choices for d_1. Then we pick an odd number of digits among $d_2, d_3, \cdots, d_n$ to be 1. The rest of them can be anything except 1, which has 9 choices each. In total, the number of choices is:

$$
8 \times \left(\binom{n-1}{1} \cdot 9^{n-2} + \binom{n-1}{3} \cdot 9^{n-4} + \ldots\right)
$$

2. If $d_1 = 1$, we have only 1 choice for d_1. Then we pick an even number of digits among $d_2, d_3, \cdots, d_n$ to be 1 and the rest of them being anything except 1. In total, the number of choices is:

$$
\binom{n-1}{0} \cdot 9^{n-1} + \binom{n-1}{2} \cdot 9^{n-3} + \ldots
$$

To compute these two summations, we first consider the binomial expansion

$$
10^{n-1} = (9+1)^{n-1} = \binom{n-1}{0} \cdot 9^{n-1} + \binom{n-1}{1} \cdot 9^{n-2} + \binom{n-1}{2} \cdot 9^{n-3} + \ldots,
$$

and

$$
8^{n-1} = (9-1)^{n-1} = \binom{n-1}{0} \cdot 9^{n-1} - \binom{n-1}{1} \cdot 9^{n-2} + \binom{n-1}{2} \cdot 9^{n-3} - \ldots.
$$

Taking their sum and differences gives:

$$
10^{n-1} + 8^{n-1} = 2 \times \left(\binom{n-1}{0} \cdot 9^{n-1} + \binom{n-1}{2} \cdot 9^{n-3} + \ldots\right),
$$

and

$$
10^{n-1} - 8^{n-1} = 2 \times \left(\binom{n-1}{1} \cdot 9^{n-2} + \binom{n-1}{3} \cdot 9^{n-4} + \ldots\right).
$$

Therefore, the total number of such numbers is given by

$$8 \times \frac{10^{n-1} - 8^{n-1}}{2} + \frac{10^{n-1} + 8^{n-1}}{2} = \frac{9}{2} \cdot 10^{n-1} - \frac{7}{2} \cdot 8^{n-1}$$

Solution 3: Consider the set S of n-digit numbers $\overline{d_1 d_2 \dots d_n}$ that contains at least a digit of 1 or 2. The total number of such number can be computed as the total number of n-digit numbers $9 \times 10^{n-1}$ minus the total number of n-digit numbers without 1 or 2, which is $7 \times 8^{n-1}$. In other words, the set S contains $9 \times 10^{n-1} - 7 \times 8^{n-1}$ elements. We now show that exactly half of them have an odd number of 1.

Define $\phi : S \to S$ in the following way: for an n-digit number $\overline{d_1 d_2 \dots d_n} \in S$ where the its i-th digit is the first digit that is either 1 or 2, define $\phi(S)$ by changing the i-th digit from 1 to 2 or 2 to 1. For example, $\phi(3512) = 3522$ and $\phi(4022) = 4012$. The map ϕ is well-defined because each number in S contains at least one digit of 1 or 2. It is also clear that $\phi(\phi(x)) = x$ since applying ϕ twice will swap the first 1 or 2 twice, leaving all the digits unchanged. Therefore, ϕ is a bijective map on S.

Let $S_1, S_2 \subset S$ be the subsets that contain odd and even numbers of 1's respectively. Then since $\phi(x)$ switches one digit of 1 to 2 or 2 to 1, the number of 1 in $\phi(x)$ differs from that in x by exactly 1. This implies that ϕ maps S_1 to S_2 and vice versa. Since ϕ is bijective, we know ϕ is a bijection from S_1 to S_2. Therefore, S_1 and S_2 have the same number of elements, each containing half of S, which is precisely $\frac{1}{2}\left(9 \times 10^{n-1} - 7 \times 8^{n-1}\right)$.

Solution 4: Following the idea in Solution 1, we have a recursive formula

$$\begin{aligned} A_{n+1} &= 9A_n + B_n, \\ B_{n+1} &= A_n + 9B_n. \end{aligned}$$

This can be rewritten in matrix form as:

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix} \cdot \begin{bmatrix} A_n \\ B_n \end{bmatrix}$$

The matrix $T = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$ is self-adjoint with two eigenvalues 8 and 10. From linear algebra, this matrix is diagonalizable as

$$\begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix} = S \cdot \begin{bmatrix} 8 & 0 \\ 0 & 10 \end{bmatrix} \cdot S^{-1}.$$

By computing the eigenvectors, we can arrive at $S = S^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. We have

$$\begin{aligned} \begin{bmatrix} A_n \\ B_n \end{bmatrix} &= \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8^{n-1} & 0 \\ 0 & 10^{n-1} \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\left(9 \cdot 10^{n-1} - 7 \cdot 8^{n-1}\right) \\ \frac{1}{2}\left(9 \cdot 10^{n-1} + 7 \cdot 8^{n-1}\right) \end{bmatrix} \end{aligned}$$