NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 2

Fall 2024

Solution. Up to symmetries of the cube, there are three possible situations. The two vertices are connected by an edge, the two vertices are opposite corners of a face, or the two vertices are opposite corners of the cube. Note also that the distance between the points $(1, 1, 2)$ and $(2, 3, 0)$ is $\sqrt{(2-1)^2 + (3-1)^2 + (0-2)^2} = \sqrt{9} = 3$.

In the first case, the side length of the cube is 3 so that the volume of the cube is $3^3 = 27$.

In the second case, the diagonal of a side has length 3 so that the side length x satisfies the equation $2x^2 = 9$, and hence $x = \dfrac{3}{\sqrt{2}}$. Then the volume is $\dfrac{27}{2\sqrt{2}}$.

In the final case, the long diagonal has length 3 so that the diagonal d of a single face satisfies the equation $2d^2 = 9$ so that $d = \dfrac{3}{\sqrt{2}}$. Then the side length x satisfies the equation $2x^2 = d^2 = \dfrac{9}{2}$, and hence $x = \dfrac{3}{2}$. In this case the volume is $\dfrac{27}{8}$.

Thus the possible volumes are 27, $\dfrac{27}{2\sqrt{2}}$, and $\dfrac{27}{8}$.