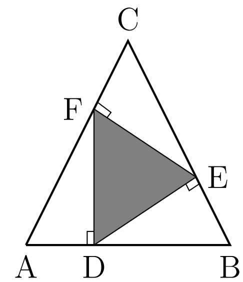
NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 4

Fall 2022

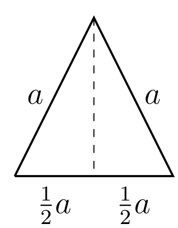
Problem. An equilateral triangle is inscribed inside another equilateral triangle such that AF = CE = DB = 2FC = 2EB = 2AD.



If the area of ΔDEF equals 9 cm^2, then find the area of ΔABC.

Solution. There are many ways of solving this problem. We will use a method that uses the area formula for an equilateral triangle.

If an equilateral triangle has sides of length a



then the height of the triangle can be calculated using the pythagorus theorem

$$\text{height} = \sqrt{a^2 - (\frac{1}{2}a)^2} = \frac{\sqrt{3}}{2}a.$$

Therefore, its area is

$$\text{Area of a triangle} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{\sqrt{3}}{4}a^2. \tag{1}$$

Since ΔDEF is an equilateral triangle, the length of each of its side equals

$$|\overline{\text{EF}}| = \sqrt{\frac{4}{\sqrt{3}}(\text{area of } \Delta\text{DEF})} = 2(3^{\frac{3}{4}})$$

Let $\ell = |\overline{\mathrm{CF}}|$. Then by assumption $|\overline{\mathrm{CE}}| = 2\ell$. Since ΔCFE is right angled at F, we apply the Pythagorus theorem to obtain

$$\begin{array}{lrcl} & |\overline{\mathrm{EF}}|^2 + |\overline{\mathrm{CF}}|^2 & = & |\overline{\mathrm{CE}}|^2 \\ \Rightarrow & (2(3^{\frac{3}{4}}))^2 + \ell^2 & = & (2\ell)^2 \\ \Rightarrow & 3\ell^2 & = & 4(3^{\frac{3}{2}}) \\ \Rightarrow & \ell^2 & = & 4(\sqrt{3}). \end{array}$$

Note that the length of each side of ΔABC is 3ℓ. Therefore by using the formula (1), we conclude

$$\begin{array}{rcl} \text{Area of a triangle } \Delta\mathrm{ABC} & = & \frac{\sqrt{3}}{4}(3\ell)^2 \\ & = & \frac{9\sqrt{3}}{4}\ell^2 \\ & = & \frac{9\sqrt{3}}{4}4(\sqrt{3}) \\ & = & 27. \end{array}$$