NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Spring 2025

Suppose **A** is a set consisting of 10 distinct two digit numbers (in the usual decimal system). Is it always possible to select two disjoint nonempty subsets of **A**, say B and C, such that the sum of all the elements in B equals the sum of all the elements in C? If your answer is yes, then provide a proof. If your answer is no, give a counterexample.

Solution. First notice it is enough to find two different subsets B' and C' of **A**, not necessarily disjoint, such that the sum of the numbers in each set are the same. This is because we can obtain two disjoint subsets satisfying the same criteria by simply removing the intersection, i.e. by setting

$$B = B' \setminus (B' \cap C') \text{ and } C = C' \setminus (B' \cap C').$$

Since all numbers have two digits, the smallest possible sum is 10 and the largest possible sum is

$$99 + 98 + \cdots + 90 = 945.$$

Thus given a set **A** of ten two digit numbers, the sum of elements of a given nonempty subset is a number between 10 and 945, i.e., there are $945 - 10 = 935$ possibilities . However, a set with 10 elements has $2^{10} - 1 = 1023$ different nonempty subsets. Thus, there are more number of subsets of **A** than the possible values for the sum of elements in the given subset. Thus, by pigeonhole principle, there must be two different subsets B' and C' of **A** which has the same sum.