NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 5

Fall 2022

Problem. How many real numbers are there which satisfy the equation

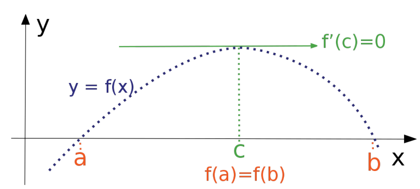
$$x^{2023} + x^{223} + x^{203} + x^{23} + x^3 + x = 2023?$$

Solution. Let $\mathfrak{F}(x) = x^{2023} + x^{223} + x^{203} + x^{23} + x^3 + x - 2023$ so that finding a solution to the equation above is equivalent to solving for $\mathfrak{F}(x) = 0$. Clearly

$$\mathfrak{F}(-10) < 0 < \mathfrak{F}(10),$$

therefore, by intermediate value theorem there is at least one value of x between -10 and 10 such that $\mathfrak{F}(x) = 0$.

If $\mathfrak{F}(x)$ have more than one zeros then there is a value of x where the tangent line of $y = \mathfrak{F}(x)$ is horizontal.



In other words, there must be a value of x where $\mathfrak{F}'(x) = 0$ (Rolle's theorem). However, the derivative

$$\mathfrak{F}'(x) = 2023x^{2022} + 223x^{222} + 203x^{202} + 23x^{22} + 3x^2 + 1$$

is a positive sum of even powers of x and a positive constant, therefore greater than zero for all values of x. So we conclude the the equation has **exactly one solution**.