# NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 5

**Spring 2023**

## Problem 5

What is the last digit of the number $2^{2023} + 7^{2023} + 8^{2023}$? Justify your answer.

**Solution.** The key here is to identify the periodic pattern of the last digits of the powers of natural numbers; the last digit of $n^{a+4}$ is the same as $n^a$, or equivalently

$$n^{a+4} - n^a \text{ is divisible by } 10$$

for any natural number $n$ when $a > 0$.

First notice that $n^{a+4} - n^a = n^a(n^4 - 1)$ is always divisible by 2; when $n$ is odd, $n^4 - 1$ is even, hence divisible by 2, and when $n$ is even, the result is true because $n^a$ is divisible by 2. Thus, it is enough to show that $n^4 - 1$ is divisible by 5 for all positive integers coprime to 5. This is a consequence of Fermat's little theorem[1].

**Theorem 1** (Fermat's little theorem)**.** *For any integer $n$ coprime to the prime $p$, $n^{p-1} - 1$ is divisible by $p$.*

Thus $2^{2023}$ has the same last digit as $2^3 = 8$, $7^{2023}$ has the same last digit as $7^3 = 343$, and $8^{2023} = 512$ has the same last digit as $8^3 = 343$. Thus the last digit of

$$2^{2023} + 7^{2023} + 8^{2023}$$

is $8 + 3 + 2 = 13$, which is 3. ■

**Fun Fact:** ***Fermat's little theorem*** *was the first of the series of theorems formulated by Fermat in the seventeenth century. Fermat did not give a proof of the* ***last theorem*** *of this series (also known as Fermat's last theorem) although he claimed to know one, citing that the margins where* ***too small to fit the proof!*** *However, Fermat's Last Theorem resisted proof, leading to doubt that Fermat ever had a correct proof. Consequently the proposition became known as a conjecture rather than a theorem. After 358 years of effort by mathematicians, the first successful proof was released in 1994 by Andrew Wiles and formally published in 1995. It was described as a "stunning advance" in the citation for Wiles's Abel Prize award in 2016.*

[1]We do not expect the knowledge of Fermat's little theorem, noticing the periodic patterns correctly to obtain the answer will lead to full points.