NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Fall 2022

Problem 8

Suppose there is a special **atom** that splits every second into two **atoms**, both identical to the original **atom**. One is placed to the right, and the other to the left, both at a unit distance from the original **atom**. However, when two **atoms** collide, they destroy each other. Thus, after two seconds there are two **atoms**, after three seconds there are four **atoms**, after four seconds there are two **atoms**, and so on. Then how many **atoms** are there after 132122 seconds?

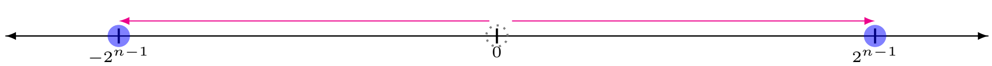
HINT: $132122 = 2^{17} + 2^{10} + 2^4 + 2^2 + 2$.

Solution. If we work out the first few steps, we notice exactly 2 **atoms** after 2, 4, 8 and 16 seconds. This is indeed a part of a general pattern.

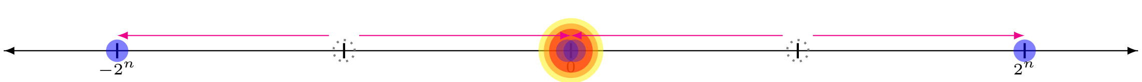
Key observation: There are exactly 2 **atoms** after 2^n seconds, where n is a positive integer.

This can be proved using the method of induction on the variable n. Suppose, there are exactly two **atoms** after 2^k seconds for $1 \leq k \leq n-1$. Then, we must show that the result is true after 2^n seconds assuming our hypothesis.

First note, after 2^{n-1} seconds we have one **atom** on each side at a distance of 2^{n-1} units from the position of the original **atom** which existed at the beginning of time (as displayed in the following numberline).

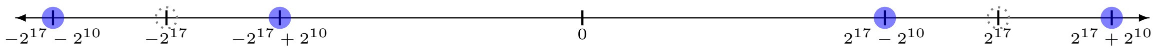


By inductive hypothesis, each of the 2 **atoms** will further produce 2 more **atoms** in the <u>next</u> 2^{n-1} seconds which will be placed at a distance of 2^{n-1} unit left and right respectively of the **atom** from which it originated from. However,



this does not lead to 4 **atoms** because there are two **atoms** are now at the origin and they annihilate each other. Thus, we have proved that after $2^{n-1} + 2^{n-1} = 2^n$ seconds, the number of **atoms** is exactly 2, completing the proof by induction.

From the key observation we conclude there are exactly 2 **atoms** after 2^{17} seconds. Each of these **atoms** convert themselves into two in the next 2^{10} seconds and all the **atoms** are at different positions. Thus, we have exactly 4 **atoms** after $2^{17} + 2^{10}$ seconds as displayed below.



Using the exact same argument we see that after $2^{17} + 2^{10} + 2^4$ seconds there are 8 **atoms**, $2^{17} + 2^{10} + 2^4 + 2^2$ there are 16 **atoms**, and after $132122 = 2^{17} + 2^{10} + 2^4 + 2^2 + 2$ there are 32 **atoms**. So 32 is the correct answer! ■

For those who are curious, there is also a general formula. For any integer N, let

$$\alpha(N) = \{\text{number of 1s in the base-2 expansion of } N\}$$

then the number of **atoms** after N seconds equals $2^{\alpha(N)}$.

For example, when $N = 132122$, its base-2 expansion is 100000010000010110 (follows from the HINT) which consists of five 1s, thus $\alpha(132122) = 5$ and number of **atoms** after 132122 seconds is $2^5 = 32$.