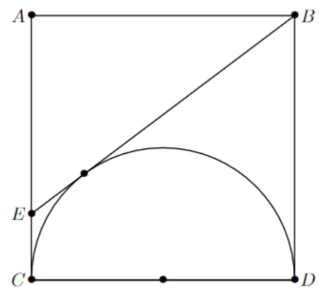
NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 2

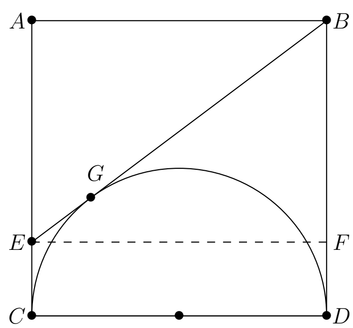
Spring 2024

Problem 2

The square $ABCD$ has side-length 8. Inside, the line BE is tangent to the semi-circle. Find the length of BE and justify your answer.



Solution 1: Let G be the point where the line BE intersects the semicircle and draw a line EF that is parallel to CD.



Notice that EG and EC are both tangent to the semicircle, and thus $EG = EC$. Similarly, $BG = BD = 8$ as they are both tangent to the semicircle as well.

Denote $EC = x$. We have that $EG = EC = x$ and thus $BE = 8 + x$, $FD = x$ and thus $BF = BD - FD = 8 - x$, and finally, $EF = 8$. Applying the Pythagorean theorem on the triangle $\triangle BEF$, we have

$$(8 + x)^2 = BE^2 = EF^2 + BF^2 = 8^2 + (8 - x)^2.$$

Expanding both sides,

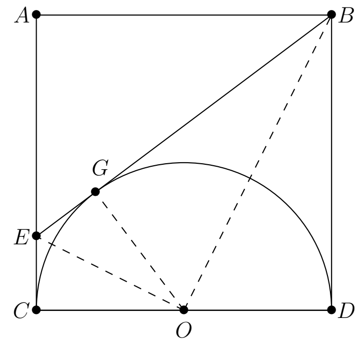
$$8^2 + 16x + x^2 = 8^2 + 8^2 - 16x + x^2.$$

This simplifies to

$$32x = 64, x = 2.$$

Therefore, $BE = 8 + x = 10$.

Solution 2: We let the the center of the semicircle be O and let BE tangent the semicircle at G.



Again, we observe that $EG = EC$ and $BG = BD$ since they are tangent to the semicircle. Moreover, $OG \perp BE$, $OC \perp EC$, $OD \perp BD$. Therefore, $\triangle EGO$ is congruent to $\triangle ECO$ and $\triangle BGO$ is congruent to $\triangle BDO$.

We have that $\angle GOE = \angle COE$ and $\angle BOG = \angle BOD = 90^o - \angle OBG$. Notice also that

$$180^o = \angle GOE + \angle COE + \angle BOG + \angle BOD = 2 \times (\angle GOE + \angle BOG).$$

Therefore, $\angle GOE = 90^o - \angle BOG = \angle OBG$. This implies that the right-angled triangle $\triangle GOE$ is similar to the right-angled triangle $\triangle GBO$. Therefore,

$$\frac{EG}{GO} = \frac{GO}{BG}.$$

Notice that $BG = BD = 8$ and $GO = OC = OD = 4$ is the radius of the semi-circle. Therefore, $EG = 2$ and thus $BE = EG + GB = 10$.