# NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 3

**Spring 2024**

Problem 3

Find all real numbers $x, y$ such that $x^2 = 8x + y$ and $y^2 = 8y + x$.

**Solution**: Taking the difference of these two equations, we have

$$x^2 - y^2 = 8x + y - (8y + x) = 7(x - y).$$

Factor the left-hand side as $x^2 - y^2 = (x - y)(x + y)$, we have that

$$(x - y)(x + y) = 7(x - y).$$

We have to be careful when we try to cancel out the $x - y$ factor. There are two cases:
**Case 1**. $x - y \neq 0$: we have $x + y = 7$ and therefore $y = 7 - x$. Replacing it into the first equation, we have

$$x^2 = 8x + (7 - x) = 7x + 7.$$

Solving these equations gives $x = \frac{7 \pm \sqrt{7^2 + 28}}{2} = \frac{7 \pm \sqrt{77}}{2}$. Using $y = 7 - x$, we have 2 sets of solutions:

$$\begin{cases} x = \frac{7+\sqrt{77}}{2} \\ y = \frac{7-\sqrt{77}}{2} \end{cases} \qquad \begin{cases} x = \frac{7-\sqrt{77}}{2} \\ y = \frac{7+\sqrt{77}}{2} \end{cases}$$

**Case 2**. $x - y = 0$: we have $x = y$ so that $x^2 = 9x$ and thus $x = 0$ or $9$. In this case, we have two other sets of solutions:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x = 9 \\ y = 9 \end{cases}$$