NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 4

Spring 2024

Problem 4

How many consecutive zeros does $2024!$ have at the right end of its decimal expansion? Here,

$$2024! = 2024 \times 2023 \times \cdots \times 2 \times 1.$$

Justify your answer.

Solution: First of all, given any integer n, the number of consecutive zeros at the right end of its decimal expansion is the largest integer k such that n is divisible by 10^k but not 10^{k+1}. To find such a number, it suffices to find the prime factorization of n as

$$n = 2^{n_2} \times 5^{n_5} \times m,$$

where m is not a multiple of 2 or 5. From here, the number of consecutive zeros at the right end of its decimal expansion is precisely $k = \min\{n_2, n_5\}$.

Therefore, to solve our question, it suffices to know the powers of 2 and 5 in the prime factorization of $2024!$. We first claim that

$$n_2 = \sum_{k \geq 1} \left\lfloor \frac{2024}{2^k} \right\rfloor = \left\lfloor \frac{2024}{2} \right\rfloor + \left\lfloor \frac{2024}{4} \right\rfloor + \cdots + \left\lfloor \frac{2024}{1024} \right\rfloor$$

Here, $\lfloor x \rfloor$ is the integer part of x. To see this formula: each even number between 1 and 2024 contributed a factor of 2 in the product $2024!$. There are precisely $\left\lfloor \frac{2024}{2} \right\rfloor$ even numbers between 1 and 2024. Among these even numbers, there are precisely $\left\lfloor \frac{2024}{4} \right\rfloor$ of them being a multiple of 4. Each of these numbers will contribute at least another factor of 2. Among these multiples of 4, there are $\left\lfloor \frac{2024}{8} \right\rfloor$ of them being a multiple of 8, which contributes another factor of 2. Repeat this process, and we get the total number of 2s in the product of $2024!$ is given by the desired formula. Another way to illustrate this formula is by considering the factor of 2 in each even number:

$$\begin{aligned}
2 &= 2 \\
4 &= 2 \times 2 \\
6 &= 2 \times 3 \\
8 &= 2 \times 2 \times 2 \\
10 &= 2 \times 5 \\
12 &= 2 \times 2 \times 3
\end{aligned}$$

$$14 = 2 \times 7$$

$$\vdots$$

The total number of 2s in the list is given by

$$n_2 = \left\lfloor \frac{2024}{2} \right\rfloor + \left\lfloor \frac{2024}{4} \right\rfloor + \left\lfloor \frac{2024}{8} \right\rfloor + \dots$$

One can compute this number explicitly as

$$n_2 = 1012 + 506 + 253 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 2017$$

Using the same logic, we can know the power of 5 in the prime factorization of $2024!$ is given by

$$n_5 = \sum_{k \geq 1} \left\lfloor \frac{2024}{5^k} \right\rfloor .$$

We can compute it explicitly as

$$\begin{aligned} n_5 &= \left\lfloor \frac{2024}{5} \right\rfloor + \left\lfloor \frac{2024}{25} \right\rfloor + \left\lfloor \frac{2024}{125} \right\rfloor + \left\lfloor \frac{2024}{625} \right\rfloor \\ &= 404 + 80 + 16 + 3 = 503 \end{aligned}$$

Therefore,

$$2024! = 2^{2017} \times 5^{503} \times m = 10^{503} \times m',$$

where m has is not a multiple of 2 or 5 and $m' = 2^{1514} m$ is not a multiple of 10. This tells us that there are 503 zeros at the right end of its decimal expansion of $2024!$.