NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 4

Spring 2024

Problem 5

Suppose four positive integers a, b, c, d satisfies

$$ab + cd = 38$$
$$ac + bd = 34$$
$$ad + bc = 43$$

Find all possible a, b, c, d. Justify your answer.

Solution: First we observe that by adding the first two equations, we get

$$ab + cd + ac + bd = 38 + 34 = 72.$$

The left-hand side can be factored, so we get

$$72 = a(b + c) + d(b + c) = (a + d)(b + c).$$

Exploring this idea, we can add any pair of these equations and get:

$$(a + d)(b + c) = 72, \quad \text{by adding equation (1) and (2),}$$
$$(a + b)(c + d) = 77, \quad \text{by adding equation (2) and (3),}$$
$$(a + c)(b + d) = 81, \quad \text{by adding equation (1) and (3),}$$

Now notice that a, b, c, d are positive integers, so that each of $a + b, a + c, a + d, b + c, b + d, c + d$ is also an integer of at least 2. Now $(a + b)(c + d) = 77 = 7 \times 11$ is a product of two primes. The only way this factorization works is when $\{a + b, c + d\}$ is $\{7, 11\}$.

There are several different ways to proceed (for example, one can discuss the cases when $a + b$ is 7 or 11 separately, and reduce the equations in two variables). We shall show one of the simpler ways from here. Let us first observe that regardless of whether $a + b = 7$ or 11 (and $c + d = 11$ or 7), we always have that $a + b + c + d = 7 + 11 = 18$. Now $81 = (a + c)(b + d)$ is factored as a product of two positive integers $(a + c)$ and $(b + d)$, whose sum is $(a + c) + (b + d) = 18$. There is only one way to factor 81 as a product of two positive integers such that their sum is 18, which is using $81 = 9 \times 9$. Therefore, we have $a + c = b + d = 9$.

Now if we subtract equation (3) from (1), we have:

$$(ab + cd) - (ad + bc) = -5.$$

The left-hand side can be factored:

$$-5 = a(b - d) - c(b - d) = (a - c)(b - d).$$

Here, $a-c$ and $b-d$ are integers (possibly negative). There are 4 different possibilities:

$$\begin{cases} a-c=1 \\ b-d=-5 \end{cases} \quad \begin{cases} a-c=-1 \\ b-d=5 \end{cases} \quad \begin{cases} a-c=5 \\ b-d=-1 \end{cases} \quad \begin{cases} a-c=-5 \\ b-d=1 \end{cases}$$

Together with $a+c=b+d=9$, we have 4 solutions corresponding to each of these 4 possibilities:

$$\begin{cases} a=5 \\ b=2 \\ c=4 \\ d=7 \end{cases} \quad \begin{cases} a=4 \\ b=7 \\ c=5 \\ d=2 \end{cases} \quad \begin{cases} a=7 \\ b=4 \\ c=2 \\ d=5 \end{cases} \quad \begin{cases} a=2 \\ b=5 \\ c=7 \\ d=4 \end{cases}$$

One can easily verify that all these solutions indeed satisfy the original equations.