NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 7

Spring 2024

Problem 7

How many ways can we pick two non-empty subsets $A, B \subset \{1, 2, \ldots, n\}$ where $A \cap B = \emptyset$? Simplify and justify your answer.

Solution 1: First of all, to construct (potentially empty) A, B with $A \cap B = \emptyset$, it is the same as the following procedure: for each $i \in \{1, 2, \ldots, n\}$, we decide for the element i to be: (1) in A only, (2) in B only, (3) in neither A, B. There are precisely 3^n of such choices, each corresponding to some choice of A, B with $A \cap B = \emptyset$.

However, among these, A or B may be the empty set. Since we are only looking for non-empty A, B, we must exclude these possibilities.

When $A = \emptyset$, B can be any subset of $\{1, 2, \ldots, n\}$, which has 2^n choices. Similarly, when $B = \emptyset$, A has 2^n choices. In total, we have $2 \times 2^n - 1$ scenarios where either A or B is empty. Here, we must subtract 1 to avoid double counting the case where $A = B = \emptyset$.

Therefore, the total number of ways to write **non-empty** A, B with $A \cap B = \emptyset$ is precisely $3^n - 2 \times 2^n + 1$.

Solution 2: Since A and B are non-empty, the number of elements in A is some integer k with $1 \leq k \leq n - 1$.

Let us consider the case when A has exactly k elements. The number of ways to choose k elements from n elements is given by the binomial coefficient $\binom{n}{k}$, which is the number of choices for A. Once A is fixed, we need to pick a non-empty set B from the remaining $n - k$ elements. There are precisely $2^{n-k} - 1$ choices for non-empty B. Therefore, there are $\binom{n}{k} \times (2^{n-k} - 1)$ possibilities.

In total, the number of choices becomes

$$C = \sum_{k=1}^{n-1} \binom{n}{k} \times (2^{n-k} - 1) \tag{*}$$

To simplify this summation, we turn to the binomial expansion:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Notice that $\binom{n}{k} = \binom{n}{n-k}$ so that this can also be written as

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} = x^n + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} + 1.$$

Plugging in $x = 1$ and 2 yields:

$$2^n = 1 + \sum_{k=1}^{n-1} \binom{n}{k} + 1, \quad \text{for } x = 1,$$

and,

$$3^n = 2^n + \sum_{k=1}^{n-1} \binom{n}{k} 2^k + 1, \quad \text{for } x = 2.$$

Substitute into equation (*), we get

$$C = \sum_{k=1}^{n-1} \binom{n}{k} 2^k - \sum_{k=1}^{n-1} \binom{n}{k} = (3^n - 2^n - 1) - (2^n - 2) = 3^n - 2 \times 2^n + 1.$$