NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Spring 2024

Problem 8

Prove that for any n integers $a_1, a_2, \dots, a_n$, we can always find some consecutive terms $a_i, a_{i+1}, \dots, a_{i+k}$ ($k \geq 0$) that add up to a multiple of n. Note that a single term a_i is also considered as "consecutive terms".

Solution: Let us consider the following partial sums:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

If any of these s_k is a multiple of n, then we have a sum of k consecutive terms that add up to a multiple of n.

Otherwise, the remainder of $s_1, s_2, \dots, s_n$ module n can only be one of $1, 2, \dots, n-1$, a total of $n-1$ choices. Since we have a total of n numbers here, we must have some indices $i < j$ such that s_i and s_j have the same remainder when divided by n. In this case, $s_j - s_i = a_{i+1} + \cdots + a_j$ is a sum of some consecutive term that is a multiple of n.