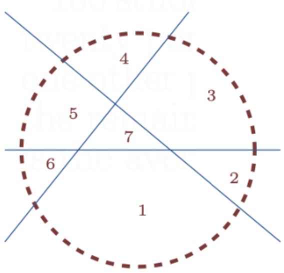
NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 9

Spring 2023

Problem 9

What is the maximum number of regions into which N straight lines (infinitely long) divides the plane ($\mathbb{R}^2$)? Justify your answer.



Example: When $\mathrm{N} = 3$ then $\#\mathrm{Regions} = 7$.

Solution. The maximum number of regions possible using N lines is given by the formula

$$\mathrm{MaxR}(\mathrm{N}) = 1 + \frac{\mathrm{N}(\mathrm{N}+1)}{2}$$

and there are two ways of solving this. The first once is an elementary induction. Suppose, we know the formula for N lines. Then the $(\mathrm{N}+1)$-th line will create an at most of $\mathrm{N}+1$ additional regions if it intersects the first N lines. Thus with $\mathrm{N}+1$ lines we can at most have

$$1 + \frac{\mathrm{N}(\mathrm{N}+1)}{2} + (\mathrm{N}+1) = 1 + \frac{(\mathrm{N}+1)(\mathrm{N}+2)}{2} = \mathrm{MaxR}(\mathrm{N}+1),$$

completing the inductive argument.

The second solution involves Euler number, which for a closed and bounded surface S is given by

$$\mathrm{EulerNumber}(\mathrm{S}) = \#\mathrm{Vertices} - \#\mathrm{Edges} + \#\mathrm{Tiles}$$

in any tessellation of S. A fascinating fact is that the Euler number is independent of how you tile your surface and depends only on the chosen surface. For example, if we consider a large disk encompassing all the intersection points of N lines drawn in $\mathbb{R}^2$, we get a tessellation of the disk, in which number of tiles equals the number of region created by N lines in $\mathbb{R}^2$. The $\mathrm{N} = 3$ case is depicted in the diagram given in the problem.

The maximum number of tiles is obtained when any two lines intersect while no three lines having any point in common. If so, we get $\binom{\mathrm{N}}{2} = \frac{\mathrm{N}(\mathrm{N}-1)}{2}$ vertices within the disk and $2\mathrm{N}$ vertices on the boundary of the disk. Further, each line is divided into N segments and the boundary of disk, which is a circle, is subdivided into 2N segments. It is also well-known that the Euler number of a disk is 1. Consequently, we get

$$\begin{aligned} 1 &= \#\mathrm{Vertices} - \#\mathrm{Edges} + \#\mathrm{Tiles} \\ &= (\frac{\mathrm{N}(\mathrm{N}-1)}{2} + 2\mathrm{N}) - (\mathrm{N}^2 + 2\mathrm{N}) + \#\mathrm{Tiles} \end{aligned}$$

which implies $\#\mathrm{Regions} = \#\mathrm{Tiles} = \mathrm{MaxR}(\mathrm{N})$ as desired. ■