NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 6

Spring 2025

Find all real solutions for x in the following equation:

$$\frac{8^x - 27^x}{12^x - 18^x} = \frac{7}{2}.$$

Solution. Observing that every x appears as a power of 2 or of 3, we begin with the substitutions $a = 2^x$ and $b = 3^x$, which gives us

$$\frac{8^x - 27^x}{12^x - 18^x} = \frac{a^3 - b^3}{a^2b - ab^2} = \frac{(a-b)(a^2+ab+b^2)}{ab(a-b)} = \frac{a^2+ab+b^2}{ab} = \frac{a}{b} + 1 + \frac{b}{a} = \frac{7}{2}.$$

Now we can apply a second substitution $t = \frac{a}{b}$ to get the equation

$$t + 1 + \frac{1}{t} = \frac{7}{2}, \text{ or equivalently } t^2 - \frac{5}{2}t + 1 = (t - \frac{1}{2})(t-2) = 0.$$

Hence we have $t = 2$ or $t = \frac{1}{2}$. Substituting back in, we have $t = \frac{a}{b} = \left(\frac{2}{3}\right)^x = \frac{1}{2}$ or 2. We conclude that either $x = \log_{2/3}(1/2)$ or $x = \log_{2/3}(2)$. Applying log rules we can rewrite this as

$$x = \frac{\pm \ln 2}{\ln 2 - \ln 3},$$

and check that indeed both are solutions to the original equation. It remains to consider the cases when $(a-b) = 0$ or $b = 0$, which have only the solution $x = 0$, which does not satisfy the original equation.