NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 9

Fall 2021

Problem 9.

Let f be a positive and continuous function that satisfies $f(x+1) = f(x)$ for every real number x. Prove the following inequality

$$\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx \geqslant 1.$$

Solution.

$$\begin{aligned}
\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx &= \int_0^{\frac{1}{2}} \frac{f(x)}{f(x+\frac{1}{2})}\,dx + \int_{\frac{1}{2}}^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx \\
&= \int_0^{\frac{1}{2}} \frac{f(x)}{f(x+\frac{1}{2})}\,dx + \int_0^{\frac{1}{2}} \frac{f(x+\frac{1}{2})}{f(x+1)}\,dx \\
&= \int_0^{\frac{1}{2}} \frac{f(x)}{f(x+\frac{1}{2})} + \frac{f(x+\frac{1}{2})}{f(x)}\,dx \\
&\geqslant \int_0^{\frac{1}{2}} \frac{1}{2}\,dx = 1,
\end{aligned}$$

where the last inequality follows by the *AM-GM inequality.*