NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 3

Spring 2021

Problem 3.

Find all integer solutions x, y, p to the equation

$$x + y + x^2 + y^2 = p$$

such that p is a prime number.

Solution.

Since $x + x^2$ and $y + y^2$ are both even, we must have p is even. Therefore,

$$p = 2.$$

Notice that

$$x + x^2 \geqslant 0 \qquad \text{and} \qquad x + x^2 > 2 \text{ if } x \neq -2, -1, 0, 1.$$

Likewise,

$$y + y^2 \geqslant 0 \qquad \text{and} \qquad y + y^2 > 2 \text{ if } y \neq -2, -1, 0, 1.$$

Thus,

$$x, y \in \{-2, -1, 0, 1\} \qquad \text{and} \quad x + y + x^2 + y^2 = 2.$$

Analyzing the possible pairs we obtain the set of all solutions (x, y):

$$\{(-2, -1),\ (-2, 0),\ (-1, -2),\ (-1, 1),\ (0, -2),\ (0, 1),\ (1, -1),\ (1, 0)\}.$$