

NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 4

Spring 2021

Problem 4.

Let p_1, \dots, p_n be a set of $n \geq 2$ points. Suppose that for any pair of points p_i and p_j for $1 \leq i < j \leq n$ there is an arrow from p_i to p_j ($p_i \rightarrow p_j$), or from p_j to p_i ($p_j \rightarrow p_i$). Prove that there is a path

$$p_{i_1} \rightarrow p_{i_2} \rightarrow \cdots \rightarrow p_{i_n}$$

that includes all of the points.

Solution.

We proceed by induction. For $n = 2$ the statement is clear.

For the induction step, assume $n \geq 3$ and assume the statement is true for any number of points $m < n$. Let G be the set of points p_i such that there is an arrow from $p_n \rightarrow p_i$, and C the set of points p_i such that there is an arrow $p_i \rightarrow p_n$. By induction hypothesis there is a path in G , and another one in C , which contain all of the points in G and C , respectively. Let p_c be the ending point of the path in C and p_g the starting point of the path in G . Then we can build a path including all of the n points by adding

$$p_c \rightarrow p_n \rightarrow p_g,$$

which completes the proof.