NMSU MATH PROBLEM OF THE WEEK
Solution to Problem 5
Spring 2022

Problem 5.

Let a_n be the number of digits of n. Compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{a_n!}.$$

Solution.
Answer: The sum is equal to $\frac{9}{10}(e^{10} - 1)$.

There are 9 numbers with one digit $\{1, \ldots, 9\}$, 90 with two digits $\{10, \ldots, 99\}$, 900 with three digits $\{100, \ldots, 999\}$, $\cdots$ It is easy to see that for $m \geqslant 1$, the number of numbers with m digits is $9 \times 10^{m-1}$.

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{a_n!} = 9 \sum_{m=1}^{\infty} \frac{10^{m-1}}{m!} = \frac{9}{10} \sum_{m=1}^{\infty} \frac{10^m}{m!} = \frac{9}{10}(e^{10} - 1),$$

where the last equality follows from the Taylor series expansion of e^x.