Problem of the Week Solution 7

May 18, 2022

Problem. Given five non-zero vectors in $\mathbb{R}^3$, prove that at least two have a non-negative dot product.

Solution.

Recall. The dot product of two vectors v_i, v_j in $\mathbb{R}^3$ is given by $||v_i||||v_j|| \cos\theta$, where θ is the angle made by the two vectors and $||v||$ represents the magnitude of the vector v. The magnitude for a non-zero vector is strictly positive.

Let v_1, v_2, v_3, v_4, v_5 be the five non-zero vectors.

Claim. We can eliminate the case where $v_i = cv_j$ for some $c \neq 0$.

Proof. Suppose that $v_i = cv_j$ with $c \neq 0$ and that all dot products are negative. If $c > 0$, then $v_i \cdot v_j = cv_j \cdot v_j = c||v_j||^2 > 0$, and so v_i and v_j have a non-negative dot product, contradicting the assumption. If $c < 0$, then for any $k \neq i, j$, we have $cv_i \cdot v_k = v_j \cdot v_k$. However, because $v_i \cdot v_k < 0$ and $v_j \cdot v_k < 0$, we have $cv_i \cdot v_k > 0$, a contradiction. Therefore, we can assume that no vector is a non-zero multiple of another.

$v_i \cdot v_j$ is non-negative if and only if $u_i \cdot u_j$ is non-negative, where $u_k = \dfrac{v_k}{||v_k||}$ is the unit vector with the same direction. *(This is because the sign of the dot product depends only on the sign on* $\cos\theta$*, which is the same for both pairs of vectors.)* We can therefore reduce to the case where we consider the corresponding unit vectors. Furthermore, because of the previous Claim, we know that $u_i \neq \pm u_1$ for any $2 \leq i \leq 5$, so the orthogonal projection of u_i onto the plane normal to u_1 is non-zero for each $2 \leq i \leq 5$.

Define $w_i := u_i - (u_i \cdot u_1)u_1$ to be the orthogonal projection of u_i onto the plane normal to u_1 for $2 \leq i \leq 5$. Assume $u_1 \cdot u_j < 0$ for all $i \neq j$. Then we have, for $i \neq j$,

$$\begin{aligned} w_i \cdot w_j &= (u_i - (u_i \cdot u_1)u_1) \cdot (u_j - (u_j \cdot u_1)u_1) \\ &= u_i \cdot u_j - (u_i \cdot u_1)(u_1 \cdot u_j) - (u_j \cdot u_1)(u_1 \cdot u_i) + (u_i \cdot u_1)(u_j \cdot u_1)(u_1 \cdot u_1) \\ &= u_i \cdot u_j - (u_i \cdot u_1)(u_j \cdot u_1)(2 - ||u_1||^2) \\ &= u_i \cdot u_j - (u_i \cdot u_1)(u_j \cdot u_1) < 0 \end{aligned}$$

because $||u_1||^2 = 1$.

Consider, however, that w_2, w_3, w_4, w_5 lie in the same plane. Without loss of generality, let the vectors lie in the plane in this order, counterclockwise, with $\theta_{i,j}$ the angle between w_i and w_j. If $w_i \cdot w_j < 0$, then $\cos(\theta_{i,j}) < 0$, so $\theta_{i,j} > \frac{\pi}{2}$. However, this would imply that $\theta_{2,3} + \theta_{3,4} + \theta_{4,5} + \theta_{5,2} > 4(\frac{\pi}{2}) = 2\pi$, which is impossible (the four angles must sum to 2π).

Therefore, there must be some pair v_i, v_j with a non-negative dot product.