NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Spring 2022

Problem 8.

Let $a_1, a_2, a_3, \ldots$ be a strictly increasing sequence of positive integers. Show that the series

$$\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{a_n}$$

diverges.

Solution.

For each $q \in \mathbb{N}$ set $S_q = \displaystyle\sum_{n=1}^{q} \frac{a_{n+1} - a_n}{a_n}$. Thus, for every $p < q$ we have

$$\begin{aligned} S_p - S_q = \sum_{n=p}^{q} \frac{a_{n+1} - a_n}{a_n} &> \sum_{n=p}^{q} \frac{a_{n+1} - a_n}{a_{n+1}} \\ &> \sum_{n=p}^{q} \frac{a_{n+1} - a_n}{a_{q+1}} \\ &= \frac{a_{q+1} - a_p}{a_{q+1}} = 1 - \frac{a_p}{a_{q+1}}. \end{aligned}$$

For a fixed p and for $q \gg 0$ we have $a_{q+1} > 2a_p$, and then $\frac{a_{q+1}-a_p}{a_{q+1}} = 1 - \frac{a_p}{a_{q+1}} > \frac{1}{2}$. We conclude that $\{S_q\}_{q\in\mathbb{N}}$ is not a Cauchy sequence and then the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{a_n} = \lim_{q\to\infty} S_q$ diverges.