NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Fall 2025

Problem 8

Let a, b, c, d be any four positive integers. Is the number

$$\Delta = abcd(a^4 - b^4)(a^4 - c^4)(a^4 - d^4)(b^4 - c^4)(b^4 - d^4)(c^4 - d^4)$$

always divisible by 13? Provide a proof if it is, or a counterexample if it is not.

Solution. The answer is yes. First observe that if 13 divides either a, b, c or d, then 13 also divides Δ. Therefore, suppose that 13 does not divide a, b, c, and d. We need the following:

Claim. For any positive integer n, if 13 does not divide n, then $n^4 \pmod{13}$ is 1, 3 or 9.

Proof of claim. By the quotient-remainder theorem, $n = 13q + r$, where $1 \leq r \leq 12$. It is straightforward to check that for such an r, we have $r^4 \pmod{13}$ is 1, 3 or 9. But $n^4 \pmod{13} = r^4 \pmod{13}$, and the result follows. $\square$

Now, it is sufficient to observe that, by the pigeonhole principle, at least two of a^4, b^4, c^4, and d^4, when divided by 13, must have the same remainder (either 1, 3, or 9). But then 13 must divide the difference of those two. Therefore, 13 must divide either $a^4 - b^4$, $a^4 - c^4$, $a^4 - d^4$, $b^4 - c^4$, $b^4 - d^4$ or $c^4 - d^4$. Consequently, 13 must also divide Δ.