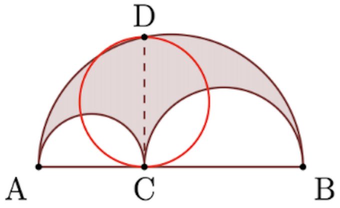
NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 6

Spring 2026

Problem 6

The shaded region in the figure below, the area inside the large semicircle but outside the two smaller ones, is known as an Arbelos. A vertical line is drawn from point **C** to the boundary of the large semicircle at point **D**.



Prove that the area of the circle with diameter **CD** is exactly equal to the area of the Arbelos.

Solution. Let the diameters of the two smaller semicircles be $2a$ and $2b$.

- The length of segment **AC** $= 2a$.
- The length of segment **CB** $= 2b$.
- The diameter of the large semicircle is **AB** $= 2a + 2b$, meaning its radius is $R = a + b$.

The Arbelos is the area of the large semicircle minus the areas of the two smaller semicircles:

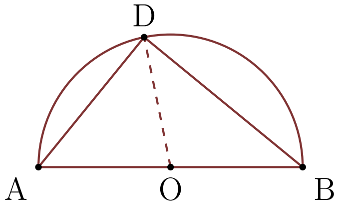
$$\begin{aligned}\text{Area}_{\text{Arbelos}} &= \frac{1}{2}\pi(a+b)^2 - \left[\frac{1}{2}\pi a^2 + \frac{1}{2}\pi b^2\right]\\ &= \frac{1}{2}\pi(a^2+2ab+b^2) - \frac{1}{2}\pi a^2 - \frac{1}{2}\pi b^2\\ &= \pi ab.\end{aligned}$$

To find the area of the red circle, we first need the length of the segment **CD**, call it h. Point **D** lies on the large semicircle. We will use the following two facts:

Fact 1: Thales's theorem
If **D** *is any point on a semicircle with diameter* **AB**, *then the triangle* $\triangle$**ADB** *is right-angled at* **D**.
Proof: Let **O** be the center of the large semicircle. Since **OA**, **OB**, and **OD** are all radii, we have

$$\mathbf{OA} = \mathbf{OB} = \mathbf{OD} = R.$$

Thus $\triangle\mathbf{AOD}$ and $\triangle\mathbf{BOD}$ are isosceles triangles



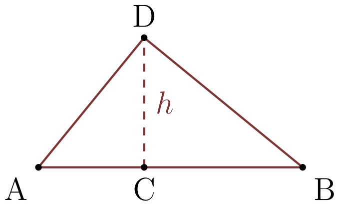
In $\triangle\mathbf{AOD}$, we let $\angle\mathbf{OAD} = \angle\mathbf{ODA} = \alpha$. Likewise, in $\triangle\mathbf{BOD}$, we set $\angle\mathbf{OBD} = \angle\mathbf{ODB} = \beta$. The sum of the angles in $\triangle\mathbf{ADB}$ must be 180°:

$$\alpha + (\alpha + \beta) + \beta = 180^\circ \implies 2\alpha + 2\beta = 180^\circ \implies \alpha + \beta = 90^\circ.$$

Since $\angle\mathbf{ADB} = \alpha + \beta$, the triangle is right-angled at $\mathbf{D}$ $\square$.

Fact 2: The Right Triangle Altitude Theorem

In a right triangle $\triangle\mathbf{ADB}$ *right-angled at* $\mathbf{D}$*, the altitude* $h = \mathbf{CD}$ *from the right angle to the hypotenuse divides the hypotenuse into two segments* $\mathbf{AC}$ *and* $\mathbf{CB}$ *such that* $h^2 = \mathbf{AC} \cdot \mathbf{CB}$*.*



Proof: In the right triangle $\triangle\mathbf{ADB}$, the altitude $\mathbf{CD}$ creates two smaller triangles, $\triangle\mathbf{ADC}$ and $\triangle\mathbf{CDB}$. These triangles are similar ($\triangle\mathbf{ADC} \sim \triangle\mathbf{CDB}$) because they share the same angles. Specifically, if $\angle\mathbf{CAD} = \theta$, then $\angle\mathbf{ADC} = 90^\circ - \theta$ and $\angle\mathbf{BDC} = \theta$. By the property of similar triangles, the ratio of corresponding sides is equal:

$$\frac{\mathbf{CD}}{\mathbf{AC}} = \frac{\mathbf{CB}}{\mathbf{CD}} \implies h^2 = \mathbf{CD}^2 = \mathbf{AC} \cdot \mathbf{CB}$$

$\square$

Proof of Fact 2 using the Pythagorean Theorem: Applying the Pythagorean theorem to the right triangles $\triangle\mathbf{ADC}$ and $\triangle\mathbf{CDB}$ yields

$$\begin{aligned} h^2 &= \mathbf{AD}^2 - \mathbf{AC}^2 \\ h^2 &= \mathbf{DB}^2 - \mathbf{CB}^2 \end{aligned}$$

Adding these two equalities gives

$$2h^2 = \mathbf{AD}^2 + \mathbf{DB}^2 - \mathbf{AC}^2 - \mathbf{CB}^2$$

Now apply the Pythagorean theorem to the right triangle $\triangle\mathbf{ADB}$ to obtain $\mathbf{AB}^2 = \mathbf{AD}^2 + \mathbf{DB}^2$. Therefore,

$$\begin{aligned} 2h^2 &= \mathbf{AB}^2 - \mathbf{AC}^2 - \mathbf{CB}^2 \\ &= (\mathbf{AC} + \mathbf{CB})^2 - \mathbf{AC}^2 - \mathbf{CB}^2 \\ &= \mathbf{AC}^2 + 2\mathbf{AC} \cdot \mathbf{CB} + \mathbf{CB}^2 - \mathbf{AC}^2 - \mathbf{CB}^2 \\ &= 2\mathbf{AC} \cdot \mathbf{CB} \end{aligned}$$

Thus, $h^2 = \mathbf{AC} \cdot \mathbf{CB}$. $\square$

Using **Fact 1** and **Fact 2**, we deduce the length of the diameter of the red circle:

$$\mathbf{CD} = h = \sqrt{\mathbf{AC} \cdot \mathbf{CB}} = \sqrt{(2a) \cdot (2b)} = \sqrt{4ab}.$$

Therefore, the radius of the red circle is $r = \sqrt{ab}$. Thus,

$$\text{Area}_{\text{Circle}} = \pi r^2 = \pi(\sqrt{ab})^2 = \pi ab = \text{Area}_{\text{Arbelos}}$$

as desired.