NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 5

Fall 2026

Problem 5

We call a 4-digit year *abcd* Split-Harmonious if $T(abcd) = ab + cd$ divides the year *abcd* evenly. For example, 2025 was Split-Harmonious because $T(2025) = 20 + 25 = 45$, and $2025/45 = 45$. However 2026 is not because $T(2026) = 46$ does not divide 2026 evenly. Find all the Split-Harmonious years in the 21st century (2000 -2099)? Justify your answer.

Solution. A year in the 21st century can be written as $2000 + cd$, where $0 \leq cd \leq 99$. Let $x = cd$. Then $2000 + cd = 2000 + x$ and it is harmonious provided $20 + x$ divides $2000 + x$ evenly. We have

$$\frac{2000 + x}{20 + x} = \frac{1980 + (20 + x)}{20 + x} = \frac{1980}{20 + x} + 1.$$

Therefore, $20 + x$ divides $2000 + x$ evenly provided $20 + x$ divides 1980 evenly. We thus need to find all x such that $0 \leq x \leq 99$ and $20 + x$ divides 1980 evenly. Breaking 1980 down into its prime factors gives

$$1980 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 11.$$

Therefore, $1980 = 11 \cdot 180$. Letting $y = 20 + x$, we have the following two cases:

Case 1: 11 does not divide y. Then 11 is not a prime factor of y, so y must divide 180. We thus have:

$$\begin{aligned}
180 = 2 \cdot y &\implies y = 90 \implies x = 70 \\
180 = 3 \cdot y &\implies y = 60 \implies x = 40 \\
180 = 4 \cdot y &\implies y = 45 \implies x = 25 \\
180 = 5 \cdot y &\implies y = 36 \implies x = 16 \\
180 = 6 \cdot y &\implies y = 30 \implies x = 10 \\
180 = 9 \cdot y &\implies y = 20 \implies x = 0.
\end{aligned}$$

Since $x \geq 0$, we need not look at any other divisor of 180 bigger than 9.

Case 2: 11 divides y. Then $y = 11 \cdot q$ and q must divide 180 (since $y = 11 \cdot q$ divides $1980 = 11 \cdot 180$). We thus have:

$$
\begin{array}{lllll}
q = 1 & \Longrightarrow & y = 11 & \Longrightarrow & x < 0, \text{ which is impossible} \\
q = 2 & \Longrightarrow & y = 22 & \Longrightarrow & x = 2 \\
q = 3 & \Longrightarrow & y = 33 & \Longrightarrow & x = 13 \\
q = 4 & \Longrightarrow & y = 44 & \Longrightarrow & x = 24 \\
q = 5 & \Longrightarrow & y = 55 & \Longrightarrow & x = 35 \\
q = 6 & \Longrightarrow & y = 66 & \Longrightarrow & x = 46 \\
q = 7 & \Longrightarrow & \text{impossible since 7 does not divide 180} & & \\
q = 8 & \Longrightarrow & \text{impossible since 8 does not divide 180} & & \\
q = 9 & \Longrightarrow & y = 99 & \Longrightarrow & x = 79 \\
q = 10 & \Longrightarrow & y = 110 & \Longrightarrow & x = 90
\end{array}
$$

Since $x \leq 99$, we need not consider larger values of q. Consequently, the harmonious years in the 21st century are:

$$
\begin{aligned}
x = 0 &\Longrightarrow 2000 + x = 2000 \\
x = 2 &\Longrightarrow 2000 + x = 2002 \\
x = 10 &\Longrightarrow 2000 + x = 2010 \\
x = 13 &\Longrightarrow 2000 + x = 2013 \\
x = 16 &\Longrightarrow 2000 + x = 2016 \\
x = 24 &\Longrightarrow 2000 + x = 2024 \\
x = 25 &\Longrightarrow 2000 + x = 2025 \\
x = 35 &\Longrightarrow 2000 + x = 2035 \\
x = 40 &\Longrightarrow 2000 + x = 2040 \\
x = 46 &\Longrightarrow 2000 + x = 2046 \\
x = 70 &\Longrightarrow 2000 + x = 2070 \\
x = 79 &\Longrightarrow 2000 + x = 2079 \\
x = 90 &\Longrightarrow 2000 + x = 2090.
\end{aligned}
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