NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 6

Fall 2026

Problem 6

Which of the following functions grows the fastest as $n \to \infty$:

$$f(n) = n^2, \quad g(n) = 2^n, \quad h(n) = (n+1)!$$

Justify your answer.

Solution. We first show that $f(n) < g(n)$ for all $n > 4$. The proof is by Mathematical Induction.

Base case: If $n = 5$, then $f(n) = 5^2 = 25$ and $g(n) = 2^5 = 32$, so $f(n) < g(n)$.
Induction step: Suppose $f(n) < g(n)$ for $n > 4$. Then

$$f(n+1) = (n+1)^2 = n^2 + 2n + 1.$$

By the Induction Hypothesis, $n^2 < 2^n$. Therefore, $f(n+1) < 2^n + 2n + 1$.

Claim: $2n + 1 < 2^n$ for all $n > 2$.
Proof: Again the proof is by induction on $n > 2$.
Base case: If $n = 3$, then $2n + 1 = 7 < 8 = 2^3 = 2^n$.
Induction step: Suppose $2n + 1 < 2^n$ for $n > 2$. Then, using the Induction Hypothesis,

$$2(n+1) + 1 = (2n+1) + 2 < 2^n + 2 < 2^n + 2^n = 2^{n+1}.$$

Thus, by Mathematical Induction, $2n + 1 < 2^n$ for all $n > 2$. □

Returning to $f(n)$ and $g(n)$, since $n > 4$, by the above claim we have

$$f(n+1) < 2^n + 2n + 1 < 2^n + 2^n = 2^{n+1} = g(n+1).$$

Consequently, by Mathematical Induction, $f(n) < g(n)$ for all $n > 4$, and hence $g(n)$ grows faster than $f(n)$ as $n \to \infty$.

We next show that $g(n) < h(n)$ for all $n > 1$. Again, the proof is by Mathematical Induction.

Base case: If $n = 2$, then $g(n) = 2^2 = 4$ and $h(n) = (2 + 1)! = 6$, so $g(n) < h(n)$.
Induction step: Suppose $g(n) < h(n)$ for $n > 1$. Then $g(n+1) = 2^{n+1} = 2 \cdot 2^n$. By the Induction Hypothesis, $2^n < (n + 1)!$. Therefore,

$$g(n + 1) = 2 \cdot 2^n < 2 \cdot (n + 1)! < (n + 2) \cdot (n + 1)! = (n + 2)! = h(n + 1).$$

We conclude, by Mathematical Induction, that $g(n) < h(n)$ for $n > 1$.

Consequently, $f(n) < g(n) < h(n)$ for all $n > 4$, and hence $h(n)$ grows the fastest as $n \to \infty$. $\square$