Physical Meaning Often Leads to Natural Derivations in Elementary Mathematics: On the Examples of Solving Quadratic and Cubic Equations

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Formulation of the problem. Derivations of many formulas or elementary mathematics – a formula for solving quadratic equations is a good example – are usually presented as useful tricks, tricks resulting from insights of ancient geniuses. The fact that these derivations do not naturally follow from the formulation of the problem makes these derivations – and thus, the resulting formulas – difficult to remember.

From this viewpoint, it is desirable to come up with natural – or at least more natural – derivation of such formulas. In this talk, we show that taking into account physical meaning can help.

Solving quadratic equations. It is straightforward to solve simple quadratic equations, of the type $a \cdot x^2 + c = 0$: similarly to linear equations, we get $x^2 = -c/a$ and then extract the square root. But what is we have a generic equation $a \cdot x^2 + b \cdot x + c = 0$?

This happens, e.g., when we know the area $s = x \cdot y$ and the perimeter $p = 2x + 2y$ of a rectangular region: we can plug in $y = p/2 - x$ into $s = x \cdot y$ and get a generic quadratic equation.

In real-life applications, x is often a numerical value of some physical quantity, and numerical values depend on the choice of the measuring unit and on the choice of a starting point. So, a natural idea is to select a new scale in which the equation would get a simpler form.

If we replace a measuring unit by a new one which is λ times larger, we get new value x' for which $x = \lambda \cdot x'$. Substituting this expression instead of x into the equation, we get a new equation $a \cdot (\lambda \cdot x)^2 + b \cdot \lambda \cdot x + c = 0$ – but it does not help to solve it.

On the other hand, selecting a new starting point changes the original numerical value x to the new value x' for which $x = x' + x_0$. Substituting this expression for x into the given quadratic equation, we get

$$a \cdot (x' + x_0)^2 + b \cdot (x + x_0) + c = a \cdot (x')^2 + (2a \cdot x_0 + b) \cdot x' + (a \cdot x_0^2 + b \cdot x_0 + c) = 0.$$

For an appropriate x_0, we can make the coefficient at x' equal to 0, and thus, get a simple quadratic equation – which we know how to solve.

Cubic equations. For a general cubic equation $a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0$, we can use the same idea as for the quadratic equation, and get a simplified equation $a \cdot (x')^3 + c' \cdot x' + d' = 0$. In contrast to the quadratic case, however, the resulting equation is still difficult to solve, we need to reduce it to an easy-to-solve case with no linear terms.

For this purpose, let us take into account that often, a quantity is the sum of several ones: e.g., the mass of a complex system is the sum of the masses of its components. Let us consider the simplest case $x' = u + v$. Substituting this expression into the above equations, we get

$$a \cdot (u+v)^3 + c' \cdot (u+v) + d' = a \cdot (u^3+v^3) + 3a \cdot u \cdot v \cdot (u+v) + c' \cdot (u+v) + d' = a \cdot (u^3+v^3) + (3 \cdot a \cdot u \cdot v + c') \cdot (u+v) + d' = 0.$$

When $3a \cdot u \cdot v = -c'$, we get the desired reduction, so $u^3 + v^3 = -d'/a$. So, we need to select u and v for which $u^3 + v^3 = -d'/a$ and $u \cdot v = -c'$ – i.e. $u^3 \cdot v^3 = (-d'/a')^3$. We know the sum and product of u^3 and v^3, so we can find both – and thus, we can find u, v, and finally $x' = u + v$.