Towards Decision Making Under Interval Uncertainty

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Formulation of the problem. In many real-life situations, we need to make a decision. The quality of the decision usually depends on the value of some quantity x. For example, in construction, the speed with which the cement hardens depends on the humidity, and thus, the proportions of the best cement mix depend on the humidity.

In practice, we often do not know the exact value of the corresponding quantity. For example, in the case of the pavement, while we can accurate measure the current humidity, what is really important is the humidity in the next few hours. For this future value, at best, we only know the bounds, i.e., we only know the interval $[\underline{x}, \overline{x}]$ that contains the actual (unknown) value x. To select a decision, we need to select some value x_0 from this interval. Which value should be select?

Our solution. In such situations of interval uncertainty, the ideal case is when the selected value x_0 is exactly equal to the actual value x. When these two values differ, i.e., when $x < x_0$ or $x > x_0$, the situation becomes worse. In both cases when $x < x_0$ and when $x > x_0$, we have losses, but we often have two different reasons for a loss. For example, if the humidity will be larger than expected, the hardening of the cement will take longer and we will lose time (and thus, money). In contrast, if the humidity is lower than expected, the cement will harden too fast, and the pavement will not be as stiff as it could be – so we will not get a premium for a good quality road (and we may even be required to repave some road segments). In both cases, the larger the difference $|x - x_0|$, the larger the loss.

The interval $[\underline{x}, \overline{x}]$ is usually reasonable narrow, so the difference is small. In this case, the dependence of the loss on the difference can be well approximated by a linear expression. So, when $x < x_0$, the loss is $\alpha_- \cdot (x_0 - x)$ for some α_-, and when $x > x_0$, the loss is $\alpha_+ \cdot (x - x_0)$ for some α_+. In the first case, the worst-case loss is when x is the smallest: $\alpha_- \cdot (x_0 - \underline{x})$; in the second case, the worst-case loss is when x is the largest: $\alpha_+ \cdot (\overline{x} - x_0)$. In general, the worst-case loss is the largest of these two: $w(x_0) = \max(\alpha_- \cdot (x_0 - \underline{x}), \alpha_+ \cdot (\overline{x} - x_0))$. The best-case loss is 0 – when we guessed the value x correctly.

In this case, all we know is that the loss is somewhere between 0 and $w(x_0)$, i.e., the gain is somewhere between $\underline{g} = -w(x_0)$ and $\overline{g} = 0$. In such situations, decision theory (see, e.g., [1]) recommends to use Hurwicz optimism-pessimism criterion, i.e.. to select some value $\alpha > 0$ and then to select an alternative for which the value $g \stackrel{\text{def}}{=} \alpha \cdot \overline{g} + (1 - \alpha) \cdot \underline{g}$ is the largest possible. In our case, $g = -\alpha \cdot w(x_0)$, so maximizing g simply means selecting the value x_0 for which $w(x_0)$ is the smallest.

Here, the value $\alpha_- \cdot (x_0 - \underline{x})$ increases with x_0, while the value $\alpha_+ \cdot (\overline{x} - x_0)$ decreases with x_0. Thus, the function $w(x_0)$ – which is the minimum of these two expressions – decreases until the point $\widetilde{x}$ at which these two expressions coincide, and then increases. So, the minimum of the worst-case loss $w(x_0)$ is attained at the point $\widetilde{x}$ for which $\alpha_- \cdot (\widetilde{x} - \underline{x}) = \alpha_+ \cdot (\overline{x} - \widetilde{x})$, i.e., for $\widetilde{x} = \widetilde{\alpha} \cdot \overline{x} + (1 - \widetilde{\alpha}) \cdot \underline{x}$, where we denoted $\widetilde{\alpha} \stackrel{\text{def}}{=} \alpha_+/(\alpha_+ + \alpha_-)$.

Comment. Interestingly, we get the same expression as with the Hurwicz criterion!

References

[1] R. D. Luce and R. Raiffa, *Games and Decisions: Introduction and Critical Survey*, Dover, New York, 1989.