In Practice, Estimates Based on Gaussian Uncertainty Are More Accurate Than Interval Estimates

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How can we describe measurement uncertainty: probabilistic approach. In many practical situations, the measurement error is caused by many independent small factors. It is known that in this case, according to the Central Limit Theorem, the resulting probability distribution is close to Gaussian (normal). It is therefore reasonable to describe measurement errors as normally distributed random variables.

A normal distribution is uniquely determined by its mean m and its standard deviation σ. Both can be estimated based on a few test measurements. Once we know the mean (known as *bias*), we can subtract it from all measurement results and conclude that the mean value of the resulting measurement error is 0.

To increase accuracy, a natural idea is to perform several (n) measurements and take the arithmetic average, then the standard deviation of the resulting estimate is $\sigma/\sqrt{n}$.

Interval uncertainty. Strictly speaking, for a normal distribution, any value is possible – just probabilities of large values are very small. In practice, we ignore these probabilities and assume that the absolute value of the measurement error is always smaller than $\Delta \stackrel{\text{def}}{=} k \cdot \sigma$, where $k = 2$, 3, or 6 – i.e., in effect, that the probability distribution is limited to the interval $[-\Delta, \Delta]$. In this case, after the measurement results in a value $\widetilde{x}$, we conclude that the actual (unknown) value x is in the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$. When we measure several times, we conclude that x is in the intersection of the corresponding intervals.

Interestingly, the large n, the width of this intersection interval decreases as $1/n$ [1], much faster than the k-σ interval corresponding to $\sigma/\sqrt{n}$ whose width decreases much slower – as $1/\sqrt{n}$.

A natural question and what we do in this talk. A natural question that we answer in this talk is: Which estimates are better in practice, for realistic values n?

Analysis of the problem and the resulting answer. According to [1], if we want the bound of the intersection interval with confidence p_0 for some $p_0 \approx 1$, we get the bound which is asymptotically equal to $\dfrac{A}{n}$, where $A = -\dfrac{2}{\rho} \cdot \ln\left(\dfrac{1 - p_0}{2}\right)$ and ρ is the probability density at the point Δ. For statistical estimate, the bound of the resulting k-σ interval is equal to $\dfrac{\Delta}{\sqrt{n}}$. These values become equal when $\dfrac{A}{n} = \dfrac{\Delta}{\sqrt{n}}$, i.e., when $n = \left(\dfrac{A}{\Delta}\right)^2$. For smaller n, the Gaussian interval is narrower.

For the k-σ interval, we have $\rho = \dfrac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\dfrac{k^2}{2}\right)$, so $A/\Delta = -2 \cdot \sqrt{2\pi} \cdot k \cdot \exp(k^2/2) \cdot \ln((1 - p_0)/2))$. In particular, for $k = 2$, when $p_0 = 0.95$, we get $n = (A/\Delta)^2 \approx 75\,000$.

Conclusion. In all realistic cases, $n \ll 75\,000$, so the Gaussian estimate is still better.

[1] G. W. Walster and V. Kreinovich, "For unknown–but–bounded errors, interval estimates are often better than averaging", *ACM SIGNUM Newsletter*, 1996, Vol. 31, No. 2, pp. 6–19.